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# **I do like CFD, VOL.1**

*Governing Equations and Exact Solutions*

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Katate Masatsuka

I do like CFD, VOL.1  
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Comments and suggestions are appreciated.  
Please feel free to send messages to the author.

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To Hiroaki Nishikawa  
and his family



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# Preface

CFD: Computational Fluid Dynamics.

I like CFD. I like it so much that I decided to write how much I like CFD. Here, in this volume, I focus on basic notations and formulas, governing equations, and exact solutions used in CFD. I hope that you like them too.

- **Basics:**

Some basic stuff such as notations, formulas, and theorems.

- **Model Equations:**

Model equations commonly used in the algorithm development.

- **Euler Equations:**

The Euler equations and related equations for inviscid flows.

- **Navier-Stokes Equations:**

The Navier-Stokes equations and related equations for viscous flows.

- **Turbulence Equations:**

Averaged Navier-Stokes equations and a little about turbulence models.

- **Exact Solutions I:**

General solutions and techniques for deriving exact solutions.

- **Exact Solutions II:**

Exact solutions for selected equations which can be used for accuracy study.

Naturally, the content is highly biased. This is everything about what I like in CFD. It does not contain any topics that I don't like. So, this is certainly not a textbook, but can still be a good reference book. In particular, this volume may be useful for students studying basics of CFD and researchers developing fundamental CFD algorithms (I hope that they like CFD too).

Katate Masatsuka  
Yorktown, February 2009