Derivation of Optimal Length Scale for Hyperbolic Diffusion Scheme

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Abstract

This note gives a detailed derivation of the optimal length scale for the hyperbolic diffusion scheme in Ref.[1]

1 Problem Statement

In Ref.[1], the following Fourier transformed first-order hyperbolic diffusion scheme was obtained on a Cartesian grid:

\[
\frac{dU_0}{dt} = MU_0,
\]

where \( M \) is given for smooth components as

\[
M = \begin{bmatrix}
-\frac{\nu \beta^2}{2hL_r} & \frac{i \nu \beta_x}{h} & \frac{i \nu \beta_y}{h} \\
\frac{i \nu \beta_x}{h L_r^2} & -\frac{\nu}{L_r^2} & 0 \\
\frac{i \nu \beta_y}{h L_r^2} & 0 & -\frac{\nu}{L_r^2}
\end{bmatrix},
\]

where \( \beta^2 = \beta_x^2 + \beta_y^2, h = L/N, L \) is a reference length scale (e.g., the domain size) that is resolved by \( N \) cells, and \( N \) is the number of grid spacings (i.e., the grid has \( N + 1 \) nodes). The eigenvalues are given by

\[
-\frac{\nu}{L_r^2}, \quad -\frac{\nu}{2L_r^2} \left( 1 + \frac{\beta^2}{2h} \pm \sqrt{1 - \frac{L_r \beta^2}{h} + \frac{L_r^2 \beta^2}{4h^2} (\beta^2 - 16)} \right).
\]

An optimal length scale was derived in Ref.[1] by requiring the expression inside the square root to be nonpositive, so that the Fourier mode propagates rather than is purely damped. The derivation is presented in the next section.

2 Derivation

We seek \( L_r \) that satisfies

\[
1 - \frac{L_r \beta^2}{h} + \frac{L_r^2 \beta^2}{4h^2} (\beta^2 - 16) \leq 0,
\]

which we write

\[
f(L_r) \leq 0, \quad f(L_r) = 1 - \frac{L_r \beta^2}{h} + \frac{L_r^2 \beta^2}{4h^2} (\beta^2 - 16).
\]

The function \( f(L_r) \) can be factored as

\[
f(L_r) = \frac{1}{4h^2} [(\beta - 4)\beta L_r - 2h] [(\beta + 4)\beta L_r - 2h].
\]
Solving \( f(L_r) = 0 \) for \( L_r \), we obtain
\[
L_r = \frac{2h}{(\beta - 4)\beta}, \quad \frac{2h}{(\beta + 4)\beta}. \tag{7}
\]
For smooth components, e.g., \( (\beta_x, \beta_y) \in (-\pi/2, \pi/2) \times (-\pi/2, \pi/2) \), we have
\[
\beta < 4, \tag{8}
\]
and therefore,
\[
\frac{2h}{(\beta - 4)\beta} < 0 < \frac{2h}{(\beta + 4)\beta}. \tag{9}
\]
Hence, we have \( f(L_r) \leq 0 \) for
\[
L_r \leq \frac{2h}{(\beta - 4)\beta} \leq 0, \quad L_r \geq \frac{2h}{(\beta + 4)\beta} \geq 0, \tag{10}
\]
but since \( L_r > 0 \), we are left with
\[
L_r \geq \frac{2h}{(\beta + 4)\beta}, \tag{11}
\]
(which indicates that the inequality sign in Ref. [1] is actually wrong). Since \( \beta \) varies from \( \beta_{\text{min}} = \pi h/L = \pi/N \) to \( \beta_{\text{max}} = \pi \), the condition is satisfied for all values of \( \beta \) if we set
\[
L_r = \frac{2h}{(\pi h/L + 4)(\pi h/L)} = \frac{2}{(\pi/N + 4)\pi} L \geq \frac{2h}{(\beta + 4)\beta}, \quad \text{for} \quad \beta \geq \pi/N. \tag{12}
\]
For small \( h \), or equivalently for large \( N \), we can set
\[
L_r = \frac{L}{2\pi} \geq \frac{2}{(\pi/N + 4)\pi} L \geq \frac{2h}{(\beta + 4)\beta}, \quad \text{for} \quad \beta \geq \pi/N. \tag{13}
\]
Note: By assumption, the length \( L \) is taken to be a reference length resolved by a sufficient number of cells. If the target equation has been nondimensionalized by a reference length scale \( L \), the length scale \( L_r \) is set as \( L_r/L = 1/(2\pi) \).

References