On Hyperbolic DG Discretization

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1 Conventional DG Discretization for Hyperbolic Diffusion

Consider the hyperbolic diffusion system:

$$\frac{\partial \mathbf{u}}{\partial \tau} + \text{div} \mathbf{F} = \mathbf{S},$$

(1)

where

$$\mathbf{u} = \begin{bmatrix} u \\ p \\ q \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} -\nu p & -\nu q \\ -u/T_r & 0 \\ 0 & -u/T_r \end{bmatrix}, \quad \mathbf{s} = \begin{bmatrix} 0 \\ -p/T_r \\ -q/T_r \end{bmatrix}.$$  

(2)

Numerical solution $\mathbf{u}_h$ is defined within an element $T_j$ as a polynomial of degree $p$:

$$\mathbf{u}_h = \sum_k \mathbf{u}_j^{(k)} \phi_j^{(k)}(x,y),$$

(3)

where $\mathbf{u}_j^{(k)}$ is the $k$-th vector of unknown coefficients, and $\phi_j^{(k)}(x,y)$ is the $k$-th polynomial basis: e.g., in the $P_1$ case,

$$\mathbf{u}_j^{(0)} = \begin{bmatrix} \bar{u} \\ \bar{p} \\ \bar{q} \end{bmatrix}, \mathbf{u}_j^{(1)} = \begin{bmatrix} u_x \\ p_x \\ q_x \end{bmatrix}, \mathbf{u}_j^{(2)} = \begin{bmatrix} u_y \\ p_y \\ q_y \end{bmatrix}.$$  

(4)

For our purpose, it is convenient to express the numerical solution in the form:

$$\mathbf{u}_h = \mathbf{B}_1 \mathbf{U}_1,$$

(5)

where, again, in the $P_1$ case,

$$\mathbf{B}_1 = \begin{bmatrix} 1 & 0 & 0 & \phi_j^{(1)} & 0 & 0 & \phi_j^{(2)} & 0 & 0 \\ 0 & 1 & 0 & 0 & \phi_j^{(1)} & 0 & 0 & \phi_j^{(2)} & 0 \\ 0 & 0 & 1 & 0 & 0 & \phi_j^{(1)} & 0 & 0 & \phi_j^{(2)} \end{bmatrix}, \quad \mathbf{U}_1 = [\bar{u}, \bar{p}, \bar{q}, u_x, p_x, q_x, u_y, p_y, q_y]^T.$$  

(6)
In the general case of $P_k$ approximation, we have
\[ \mathbf{u}_k = \mathbf{B}_k \mathbf{U}_k, \quad (7) \]
where
\[ \mathbf{B}_k = \begin{bmatrix} \mathbf{I} & \phi_j^{(1)} \mathbf{I} & \phi_j^{(2)} \mathbf{I} & \cdots & \phi_j^{(n)} \mathbf{I} \end{bmatrix}, \quad \mathbf{U}_k = \begin{bmatrix} \mathbf{u}_j^{(0)} \mathbf{T} & \mathbf{u}_j^{(1)} \mathbf{T} & \mathbf{u}_j^{(2)} \mathbf{T} & \cdots & \mathbf{u}_j^{(n)} \mathbf{T} \end{bmatrix}, \quad n = \frac{k(k+3)}{2}, \quad (8) \]
The DG discretization is obtained by
\[ \int_{T_j} \mathbf{B}_1^T \frac{\partial \mathbf{u}_h}{\partial \tau} \, dV + \int_{T_j} \mathbf{B}_1^T \text{div} \mathbf{F} \, dV = \int_{T_j} \mathbf{B}_1^T \mathbf{S} \, dV, \quad (9) \]
which becomes by integration by parts
\[ \mathbf{M}_1 \frac{\partial \mathbf{U}_1}{\partial \tau} + \oint_{\partial T_j} \mathbf{B}_1^T \mathbf{F} n \, dS - \int_{T_j} (\text{grad} \mathbf{B}_1^T) : \mathbf{F} \, dV = \int_{T_j} \mathbf{B}_1^T \mathbf{S} \, dV. \quad (10) \]
where $\mathbf{M}_1$ is the mass matrix,
\[ \mathbf{M}_1 = \int_{T_j} \mathbf{B}_1^T \mathbf{B}_1 \, dV. \quad (11) \]
This is the pseudo-time evolution equations for the vector of unknown coefficients $\mathbf{U}_1$ in the element $T_j$.
It is straightforward to extend it to higher-order polynomials. For the $P_k$ polynomial approximation,
\[ \mathbf{u}_k = \mathbf{B}_k \mathbf{U}_k, \quad (12) \]
the Galerkin discretization is obtained as
\[ \mathbf{M}_k \frac{\partial \mathbf{U}_k}{\partial \tau} + \oint_{\partial T_j} \mathbf{B}_k^T \mathbf{F} n \, dS - \int_{T_j} (\text{grad} \mathbf{B}_k^T) : \mathbf{F} \, dV = \int_{T_j} \mathbf{B}_k^T \mathbf{S} \, dV. \quad (13) \]
where $\mathbf{M}_k$ is the mass matrix,
\[ \mathbf{M}_k = \int_{T_j} \mathbf{B}_k^T \mathbf{B}_k \, dV. \quad (14) \]

2 Reduced DG Discretization for Hyperbolic Diffusion

To construct a DG method with reduced number of coefficients, we begin by defining a reduced polynomial approximation:
\[ \mathbf{u}_k = \mathbf{C}_k \mathbf{V}_k, \quad (15) \]
where $\mathbf{C}_k$ is a modified basis matrix and $\mathbf{V}_k$ is a vector of unknown coefficients such that
\[ m < n, \quad \mathbf{V}_k \in \mathbb{R}^m, \quad \mathbf{U}_k \in \mathbb{R}^n. \quad (16) \]
Once we define $C_k V_k$, the Galerkin discretization is obtained straightforwardly by multiplying the target equation by the modified basis functions $C_k$ and integrating by parts:

$$M_k \frac{\partial V_k}{\partial \tau} + \int_{\partial T_j} C_k^T F_n dS - \int_{T_j} (\text{grad} C_k^T) : F dV = \int_{T_j} C_k^T S dV.$$ (17)

where $M_k$ is the mass matrix,

$$M_k = \int_{T_j} C_k^T C_k dV.$$ (18)

This is the pseudo-time evolution equations for $V_k$ in the element $T_j$.

2.1 $P_1$ Case

In the $P_1$ case, $u_x$ and $u_y$ can be replaced by $\overline{p}$ and $\overline{q}$, respectively. Also, we can introduce a single coefficient $v_{xy}$ to represent both $p_y$ and $q_x$. That is, instead of

$$u_h = \overline{u} + u_x \phi_x + u_y \phi_y, \quad p_h = \overline{p} + p_x \phi_x + p_y \phi_y, \quad q_h = \overline{q} + q_x \phi_x + q_y \phi_y,$$

where $\phi_x = \phi_j^{(1)}$ and $\phi_y = \phi_j^{(2)}$, we define

$$u_h = \overline{u} + \overline{p} \phi_x + \overline{q} \phi_y, \quad p_h = \overline{p} + p_x \phi_x + v_{xy} \phi_y, \quad q_h = \overline{q} + v_{xy} \phi_x + q_y \phi_y,$$

which can be written (by replacing $U_1$ by $\tilde{U}_1$ in Equation (5)) as

$$u_h = B_1 \tilde{U}_1 = B_1 Z_1 V_1,$$ (25)

where

$$\tilde{U}_1 = [\overline{u}, \overline{p}, \overline{q}, p_x, p_y, v_{xy}, q_x, q_y]^T,$$

$$V_1 = [\overline{u}, \overline{p}, \overline{q}, p_x, v_{xy}, q_y]^T,$$

$$Z_1 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}.$$ (28)
Finally, the polynomial $u_h$ can be upgraded to quadratic because the second derivatives are available as $p_x, v_{xy}, q_y$:

$$u_h = \Phi + \Phi_x + \Phi_y + p_x \Phi_{xx} + v_{xy} \Phi_{xy} + q_y \Phi_{yy},$$

(29)

where $\Phi_{xx} = \Phi_j^{(3)}$, $\Phi_{xy} = \Phi_j^{(4)}$, and $\Phi_{yy} = \Phi_j^{(5)}$. The extra quadratic terms are added as follows:

$$u_h = B_1 \tilde{U}_1 + r_u c^T_1 V_1 = (B_1 Z_1 + r_u c^T_1) V_1 = C_1 V_1,$$

(30)

where $r_u$ is the vector indicating which variable is given the extra terms and $c^T_1$ is the vector of extra basis functions:

$$r_u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad c^T_1 = [0, 0, 0, \Phi_{xx}, \Phi_{xy}, \Phi_{yy}].$$

(31)

The advective part will then be upgraded to third-order accurate provided quadrature formulas are also upgraded. These extra quadratic terms are optional in $P_1$, but extra high-order terms in $u_h$ will be a requirement in the $P_2$ case and beyond for consistency as we will discuss later.

The Galerkin discretization is obtained by multiplying the target equation by the basis functions $C_1$ and integrating by parts:

$$M_1 \frac{\partial V_1}{\partial \tau} + \oint_{\partial T_j} C^T_1 F_n dS - \int_{T_j} (\nabla C^T_1) : F dV = \int_{T_j} C^T_1 S dV.$$

(32)

where $M$ is the mass matrix,

$$M_1 = \int_{T_j} C^T_1 C_1 dV.$$

(33)

This is the pseudo-time evolution equations for the vector of unknown coefficients in the element $T_j$.

### 2.2 $P_2$ Case

In the $P_2$ case, we begin with the standard quadratic polynomials:

$$u_h = \Phi + u_x \Phi_x + u_y \Phi_y + u_{xx} \Phi_{xx} + u_{xy} \Phi_{xy} + u_{yy} \Phi_{yy},$$

(34)

$$p_h = \varrho + p_x \Phi_x + p_y \Phi_y + p_{xx} \Phi_{xx} + p_{xy} \Phi_{xy} + p_{yy} \Phi_{yy},$$

(35)

$$q_h = \vartheta + q_x \Phi_x + q_y \Phi_y + q_{xx} \Phi_{xx} + q_{xy} \Phi_{xy} + q_{yy} \Phi_{yy}.$$

(36)

The coefficients $u_x$ and $u_y$ are the point values at the centroid of the gradients, and so can be replaced as

$$u_x \leftarrow p_h(x_c, y_c) = \varrho + p_{xx} \Phi_{xx}^c + p_{xy} \Phi_{xy}^c + p_{yy} \Phi_{yy}^c,$$

(37)

$$u_y \leftarrow q_h(x_c, y_c) = \vartheta + q_{xx} \Phi_{xx}^c + q_{xy} \Phi_{xy}^c + q_{yy} \Phi_{yy}^c,$$

(38)

where the superscript $c$ denotes the evaluation of the basis function at the centroid. Furthermore, some of the coefficients can be unified as

$$v_{xy} = p_y = q_x,$$

(39)

$$v_{xxy} = p_{xy} = q_{xx},$$

(40)

$$v_{xyy} = p_{yy} = q_{xy}.$$

(41)
Therefore, we obtain the reduced polynomial approximations as

\[ u_h = \mathbf{u} + (\mathbf{p} + p_{xx}\phi_x^c + v_{xxy}\phi_y^c + v_{xyy}\phi_y^c + q_{yy}\phi_y^c)\phi_x + (\mathbf{q} + v_{xxy}\phi_x^c + v_{xyy}\phi_x^c + q_{yy}\phi_y^c)\phi_y + p_x\phi_{xx} + v_{xy}\phi_{xy} + q_y\phi_{yy}, \]

\[ p_h = \mathbf{p} + p_x\phi_x + v_{xy}\phi_y + p_{xx}\phi_{xx} + v_{xxy}\phi_{xy} + v_{xyy}\phi_{yy}, \]

\[ q_h = \mathbf{q} + v_{xy}\phi_x + q_y\phi_y + v_{xxy}\phi_{xx} + v_{xyy}\phi_{xy} + q_{yy}\phi_{yy}. \]

At this point, it is important to note that \( u_h \) is not a valid polynomial approximation:

\[ u_h = \mathbf{u} + \mathbf{p}\phi_x + \mathbf{q}\phi_y + p_{xx}\phi_{xx} + v_{xy}\phi_{xy} + q_y\phi_{yy} + p_{xx}\phi_{xx}^c + v_{xxy}\phi_{xy}^c + v_{xyy}\phi_{yy}^c + q_{yy}\phi_{yy}^c, \]

which shows that the basis functions

\[ 1, \phi_x, \phi_y, \phi_{xx}, \phi_{xy}, \phi_{yy}, \phi_{xx}^c, \phi_{xy}^c, \phi_{yy}^c, \phi_{xx}^c, \phi_{xy}^c, \phi_{yy}^c, \phi_{xx}^c, \phi_{xy}^c, \phi_{yy}^c, \] are not independent, e.g., \( \phi_x \) and \( \phi_{xx}^c \phi_x \) are not independent. This leads to an underdetermined problem. For example, the Galerkin discretization of the advection equation \( \partial_t u = 0 \) with a polynomial \( u_h = a + b\phi_x + c\phi_y \) will result in two independent discrete equations for three coefficients \( (a, b, c) \). Consequently, the coefficients cannot be determined uniquely, thus leading to inconsistency. Some coefficients could be obtained accurately with others having no accuracy: e.g., \( a \) and \( b \) are accurate but \( c \) is not accurate at all. In the \( P_2 \) hyperbolic DG case, \( \mathbf{p} \) and \( \mathbf{q} \) could be accurate and \( p_{xx}, v_{xxy}, v_{xyy} \), and \( q_{yy} \) have no accuracy. Of course, there is no guarantee that this is the case. In the worst case, all coefficients have no accuracy. To resolve the issue, we add a cubic term in \( u_h \), which is possible because \( p_{xx}, v_{xxy}, v_{xyy}, \) and \( q_{yy} \) correspond to the third derivatives of \( u \):

\[ u_h = \mathbf{u} + \mathbf{p}\phi_x + \mathbf{q}\phi_y + p_{xx}\phi_{xx} + v_{xy}\phi_{xy} + q_y\phi_{yy} + p_{xx}\phi_{xx}^c + v_{xxy}\phi_{xy}^c + v_{xyy}\phi_{yy}^c + q_{yy}\phi_{yy}^c, \]

where \( \phi_{xx} = \phi_j^6, \phi_{xy} = \phi_j^7, \phi_{yy} = \phi_j^8, \) and \( \phi_{yy} = \phi_j^9, \) which becomes

\[ u_h = \mathbf{u} + \mathbf{p}\phi_x + \mathbf{q}\phi_y + p_{xx}\phi_{xx} + v_{xy}\phi_{xy} + q_y\phi_{yy} + p_{xx}\phi_{xx}^c + v_{xxy}\phi_{xy}^c + v_{xyy}\phi_{yy}^c + q_{yy}\phi_{yy}^c, \]

\[ v_{xxy}(\phi_{xyy}\phi_x^c + \phi_{xyy}\phi_y + \phi_{xyy}) + q_{yy}(\phi_{yy}\phi_y + \phi_{yy}). \]

The basis functions are now independent:

\[ 1, \phi_x, \phi_y, \phi_{xx}, \phi_{xy}, \phi_{yy}, (\phi_{xx}^c \phi_x + \phi_{xx}), (\phi_{xy}^c \phi_x + \phi_{xx}), (\phi_{xy}^c \phi_x + \phi_{xy}), (\phi_{yy}^c \phi_x + \phi_{yy}), (\phi_{yy}^c \phi_y + \phi_{xy}). \]

In this sense, the one-order-higher polynomial in \( u_h \) is not an option but a requirement in higher-order schemes. In summary, we have the following polynomial approximation in the \( P_2 \) hyperbolic DG:

\[ u_h = \mathbf{u} + \mathbf{p}\phi_x + \mathbf{q}\phi_y + p_{xx}\phi_{xx} + v_{xy}\phi_{xy} + q_y\phi_{yy} + p_{xx}\phi_{xx}^c + v_{xxy}\phi_{xy}^c + v_{xyy}\phi_{yy}^c + q_{yy}\phi_{yy}^c, \]

\[ p_h = \mathbf{p} + p_x\phi_x + v_{xy}\phi_y + p_{xx}\phi_{xx} + v_{xxy}\phi_{xy} + v_{xyy}\phi_{yy}, \]

\[ q_h = \mathbf{q} + v_{xy}\phi_x + q_y\phi_y + v_{xxy}\phi_{xx} + v_{xyy}\phi_{xy} + q_{yy}\phi_{yy}. \]
which is written as

\[
\mathbf{u}_h = \mathbf{B}_2 \bar{\mathbf{U}}_2 + \mathbf{r}_u \mathbf{c}_2^T \mathbf{V}_2 = (\mathbf{B}_2 \mathbf{Z}_2 + \mathbf{r}_u \mathbf{c}_2^T) \mathbf{V}_2 = \mathbf{C}_2 \mathbf{V}_2, \tag{53}
\]

where

\[
\mathbf{r}_u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_2^T = [0, 0, 0, 0, \phi_{xxx}, \phi_{xxxy}, \phi_{xyy}, \phi_{yxy}]. \tag{54}
\]

\[
\bar{\mathbf{U}}_2 = \begin{bmatrix} \pi \\ \bar{p} \\ \bar{q} \\ p_x + p_{xx} \phi_{xx}^c + v_{xxy} \phi_{xy}^c + v_{xyy} \phi_{yy}^c \\ p_{xx} \\ v_{xxy} \\
q + v_{xxy} \phi_{xx}^c + v_{xyy} \phi_{xy}^c + q_{yy} \phi_{yy}^c \\
v_{xxy} \\ q_y \\ p_x \\ p_{xx} \\ v_{xxy} \\ v_{xyy} \\ v_{xxy} \\ v_{xyy} \\ q_y \\ v_{xyy} \\ q_{yy} \end{bmatrix}, \quad \mathbf{V}_2 = \begin{bmatrix} \pi \\ \bar{p} \\ \bar{q} \\ p_x \\ q_y \\ p_{xx} \\ v_{xxy} \\ v_{xyy} \\ v_{xxy} \\ q_{yy} \end{bmatrix}, \tag{55}
\]
The Galekrin discretization is obtained by multiplying the target equation by the basis functions $C_2$ and integrating by parts:

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & \phi_{xx}^c & \phi_{xy}^c & \phi_{yy}^c & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
V_2 \\
F_n \\
F_d
\end{bmatrix} = \begin{bmatrix}
\int_{T_j} C_2^T C_2 dV
\end{bmatrix}

(56)

The Galekrin discretization is obtained by multiplying the target equation by the basis functions $C_2$ and integrating by parts:

$$
M_2 \frac{\partial V_2}{\partial \tau} + \int_{\partial T_j} C_2^T F_n dS - \int_{T_j} (\text{grad } C_2^T) : F dV = \int_{T_j} C_2^T S dV.
$$

(57)

where $M_2$ is the mass matrix,

$$
M_2 = \int_{T_j} C_2^T C_2 dV.
$$

(58)

This is the pseudo-time evolution equations for the vector of unknown coefficients in the element $T_j$.

### 2.3 $P_k$ Case

In the general case, the hyperbolic DG method can be constructed as follows. First, we define a reduced polynomial approximation as

$$
u_h \in \tilde{P}_{k+1}, \quad p_h \in P_k, \quad q_h \in P_k,
$$

(59)
where \( P_k \) denotes a vector space of polynomials spanned by the Taylor basis functions of degree \( k \), and \( \tilde{P}_{k+1} \) denotes a vector space of polynomials spanned by a modified Taylor basis functions of degree \( k+1 \) (see (49) for \( \tilde{P}_3 \)). Express the numerical solution in terms of a polynomial approximation in the form:

\[
\mathbf{u}_h = \mathbf{C}_k \mathbf{V}_k,
\]

(60)

where the basis function matrix \( \mathbf{C}_k \) consists of two parts: the reduction of the unknown coefficients and the addition of extra high-order terms, which are characterized by \( \mathbf{B}_k \mathbf{Z}_k \) and \( \mathbf{r} \mathbf{c}_k^T \), respectively,

\[
\mathbf{C}_k = \mathbf{B}_k \mathbf{Z}_k + \mathbf{r} \mathbf{c}_k^T.
\]

(61)

Note that \( \mathbf{B}_k \) is the baseline DG polynomial basis function matrix of degree \( k \) and it is \( \mathbf{Z}_k \) that characterizes the reduction operation. Note also that \( \mathbf{r} \) specifies the variables that are given extra high-order terms and \( \mathbf{c}_k^T \) is the vector of the extra high-order Taylor basis functions of degree \( k+1 \). Given the polynomial approximation, we perform the Galerkin discretization:

\[
\begin{align*}
\mathbf{M}_k \frac{\partial \mathbf{V}_k}{\partial \tau} + \int_{\partial T_j} \mathbf{C}_k^T \mathbf{F}_n \, dS - \int_{T_j} (\nabla \mathbf{C}_k^T) : \mathbf{F} \, dV &= \int_{T_j} \mathbf{C}_k^T \mathbf{S} \, dV. \\
\end{align*}
\]

(62)

where \( \mathbf{M}_k \) is the mass matrix,

\[
\mathbf{M}_k = \int_{T_j} \mathbf{C}_k^T \mathbf{C}_k \, dV.
\]

(63)
A Note on Hyperbolic Method and DG/RDG Methods

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1 DG, RDG, and Hyperbolic-RDG

Consider the advection-diffusion equation:

$$\partial_\tau u + \partial_x f = \nu \partial_{xx} u, \quad (1)$$

where $\tau$ is a pseudo time, $f$ is a convective flux, and $\nu$ is a positive constant diffusion coefficient. The time derivative term has been defined with a pseudo time because we are interested to develop a steady solver, which serves as a nonlinear solver required in implicit time-stepping schemes for unsteady problems.

A $P_2$ discontinuous Galerkin method based on a polynomial defined in each computational cell would require a piecewise quadratic polynomial to represent the solution $u$:

$$u_j(x) = \bar{u}_j + (\bar{u}_x)_j \phi_x + (\bar{u}_{xx})_j \phi_{xx}, \quad (2)$$

where $\phi_x$ and $\phi_{xx}$ are linear and quadratic basis functions, respectively, $\bar{u}_j$, $\bar{u}_x)_j$, and $\bar{u}_{xx})_j$ are the degrees of freedom. Therefore, the $P_2$ DG method requires three degrees of freedom per cell. The method is formally third-order accurate. On the other hand, in the reconstructed DG method (RDG), we reconstruct $(\bar{u}_{xx})_j$ from $(\bar{u}_x)_j$,

$$u_j(x) = \bar{u}_j + (\bar{u}_x)_j \phi_x + (\tilde{u}_{xx})_j \phi_{xx}, \quad (3)$$

where $(\tilde{u}_{xx})_j$ denotes a reconstructed value, and thus we need only two degrees of freedom, $\bar{u}_j$ and $(\bar{u}_x)_j$, to achieve third-order of accuracy.

In the hyperbolic method, we reformulate the diffusion term as a first-order system that is hyperbolic in the pseudo time:

$$\partial_\tau u + \partial_x f = \nu \partial_x p, \quad (4)$$

$$\partial_\tau p = \frac{1}{T_r} (\partial_x u - p), \quad (5)$$

where $T_r = L_r^2 / \nu$ and $L_r = 1/(2\pi)$; the system has been designed to recover the advection-diffusion equation (1) in the pseudo steady state. Consider a $P_0$ discontinuous Galerkin method (i.e., a cell-centered finite-volume method), which requires one degree of freedom for each variable:

$$u_j(x) = \bar{u}_j, \quad (6)$$

$$p_j(x) = \bar{p}_j. \quad (7)$$

This leads to a first-order scheme for both the advective and diffusive terms. Note, then, that as we will have $p = \partial_x u$ in the pseudo steady state, we can upgrade $u_j(x)$ to a linear polynomial:

$$u_j(x) = \bar{u}_j + \bar{p}_j \phi_x, \quad (8)$$

$$p_j(x) = \bar{p}_j. \quad (9)$$
This gives second-order accuracy for the advective term while the diffusive-term approximation remains first order accurate. Yet, employing the RDG method, we linearly reconstruct \((\tilde{\rho}_x)_j\) from \(\tilde{p}_j\) and upgrade both polynomials by one order:
\[
\begin{align*}
u_j(x) &= \tilde{u}_j + \tilde{p}_j \phi_x + (\tilde{p}_x)_j \phi_{xx}, \quad (10) \\
p_j(x) &= \tilde{p}_j + (\tilde{p}_x)_j \phi_x, \quad (11)
\end{align*}
\]
which results in third- and second-order accuracy for the advective and diffusive terms, respectively. This method is called the hyperbolic-RDG-L method. Ultimately, we may quadratically reconstruct \((\tilde{p}_x)_j\) from \(\tilde{p}_j\), so that we will have \((\tilde{p}_x)_j\) as well, and set
\[
\begin{align*}
u_j(x) &= \tilde{u}_j + \tilde{p}_j \phi_x + (\tilde{p}_x)_j \phi_{xx} + (\tilde{p}_{xx})_j \phi_{xxx}, \quad (12) \\
p_j(x) &= \tilde{p}_j + (\tilde{p}_x)_j \phi_x + (\tilde{p}_{xx})_j \phi_{xx}. \quad (13)
\end{align*}
\]
This leads to fourth- and third-order accuracy for the advective and diffusive terms, respectively. This method is called the hyperbolic-RDG-Q method. Note that the discretization is performed purely as a \(P_0\) discontinuous Galerkin method in all cases, and thus the only basis function used in the weak formulation is \(1\). The number of degrees of freedom is thus equal to that in the RDG method.

Extending the discussion, we have a fourth-order DG method with
\[
\begin{align*}
u_j(x) &= \tilde{u}_j + (\tilde{u}_x)_j \phi_x + (\tilde{u}_{xx})_j \phi_{xx} + (\tilde{u}_{xxx})_j \phi_{xxx}, \quad (14)
\end{align*}
\]
where \(\phi_{xx}\) is a cubic basis function, and \((\tilde{u}_{xxx})_j\) is the corresponding degree of freedom, and a fourth-order RDG method with
\[
\begin{align*}
u_j(x) &= \tilde{u}_j + (\tilde{u}_x)_j \phi_x + (\tilde{u}_{xx})_j \phi_{xx} + (\tilde{u}_{xxx})_j \phi_{xxx}, \quad (15)
\end{align*}
\]
where \((\tilde{u}_{xxx})_j\) denotes a reconstructed value. In the hyperbolic method, we have a \(P_1\) DG method with
\[
\begin{align*}
u_j(x) &= \tilde{u}_j + (\tilde{u}_x)_j \phi_x, \quad (16) \\
p_j(x) &= \tilde{p}_j + (\tilde{p}_x)_j \phi_x. \quad (17)
\end{align*}
\]
This leads to a second-order scheme. But again we can replace \((\tilde{u}_x)_j\) by \((\tilde{p}_x)_j\) and upgrade the polynomial of \(u\):
\[
\begin{align*}
u_j(x) &= \tilde{u}_j + (\tilde{p}_x)_j \phi_x + (\tilde{p}_{xx})_j \phi_{xx}, \quad (18) \\
p_j(x) &= \tilde{p}_j + (\tilde{p}_x)_j \phi_x, \quad (19)
\end{align*}
\]
to obtain third-order accuracy in the advective term (and second-order accuracy in the diffusive term). Using the RDG method with a linear reconstruction (Hyperbolic-RDG-L), we obtain
\[
\begin{align*}
u_j(x) &= \tilde{u}_j + (\tilde{p}_x)_j \phi_x + (\tilde{p}_{xx})_j \phi_{xx} + (\tilde{p}_{xxx})_j \phi_{xxx}, \quad (20) \\
p_j(x) &= \tilde{p}_j + (\tilde{p}_x)_j \phi_x + (\tilde{p}_{xx})_j \phi_{xx} + (\tilde{p}_{xxx})_j \phi_{xxx}, \quad (21)
\end{align*}
\]
which gives fourth- and third-order accuracy for the advective and diffusive terms, respectively. Finally, applying a quadratic reconstruction (Hyperbolic-RDG-Q), we obtain
\[
\begin{align*}
u_j(x) &= \tilde{u}_j + (\tilde{p}_x)_j \phi_x + (\tilde{p}_{xx})_j \phi_{xx} + (\tilde{p}_{xxx})_j \phi_{xxx} + (\tilde{p}_{xxxx})_j \phi_{xxxx}, \quad (22) \\
p_j(x) &= \tilde{p}_j + (\tilde{p}_x)_j \phi_x + (\tilde{p}_{xx})_j \phi_{xx} + (\tilde{p}_{xxx})_j \phi_{xxx} + (\tilde{p}_{xxxx})_j \phi_{xxxx}, \quad (23)
\end{align*}
\]
which leads to 5th- and 4th-order accuracy for the advective and diffusive terms, respectively.

Potential advantages of the hyperbolic DG and RDG methods are

1. Simplicity in the discretization, upwind fluxes for all terms.
2. Simple and systematic construction of efficient iterative solvers.
3. The numerical stiffness associated with the second derivative is completely eliminated, implying convergence acceleration for problems where diffusion is important.
4. The quality of the solution gradient on highly irregular stretched grids can be greatly improved in the hyperbolic method while conventional methods are known to generate oscillations.

Accuracy on irregular grids is very important for grid adaptation, which can easily result in ‘bad’ grids especially for viscous problems. Potential applications can be found in any viscous flow problem for both the incompressible and compressible Navier-Stokes equations. A suitable hyperbolic viscous system is already available.
2 Comparison of Degrees of Freedom

In two dimensions, again for a scalar equation, we consider the advection-diffusion equation:

\[
\partial_x u + \partial_y f + \partial_y g = \nu (\partial_{xx} u + \partial_{yy} u),
\]

where \( g \) is a flux in \( y \)-direction. The \( P_2 \) DG method requires a quadratic polynomial in a computational cell:

\[
u_j(x, y) = u_j + (u_x)_j \phi_x + (u_y)_j \phi_y + (u_{xx})_j \phi_{xx} + (u_{xy})_j \phi_{xy} + (u_{yy})_j \phi_{yy},
\]

where \( \phi_x \) and \( \phi_y \) are linear and quadratic basis functions in \( y \), and \( \phi_{xy} \) is a basis function of a cross term. The polynomial involves six degrees of freedom. The method is formally third-order accurate. On the other hand, the RDG method requires only a linear polynomial, with reconstructed quadratic terms, to achieve third-order accuracy:

\[
u_j(x, y) = u_j + (u_x)_j \phi_x + (u_y)_j \phi_y + (u_{xx})_j \phi_{xx} + (u_{xy})_j \phi_{xy} + (u_{yy})_j \phi_{yy},
\]

and thus only three degrees of freedom are required. In the hyperbolic method, we discretize the following system:

\[
\partial_x u + \partial_y f + \partial_y g = \nu (\partial_x p + \partial_y q),
\]

\[
\partial_x p = \frac{1}{T_r} (\partial_x u - p),
\]

\[
\partial_y q = \frac{1}{T_r} (\partial_y u - q).
\]

As in the 1D case, it suffices to have a piecewise constant representation for each variable:

\[
u_j(x, y) = \overline{u}_j,
\]

\[
p_j(x, y) = \overline{p}_j,
\]

\[
q_j(x, y) = \overline{q}_j,
\]

to achieve third-order accuracy in the advective term because we can upgrade the polynomials, by reconstruction, as follows:

\[
u_j(x, y) = \overline{u}_j + \overline{p}_j \phi_x + \overline{q}_j \phi_y + (\overline{p}_x)_j \phi_{xx} + (\overline{p}_y)_j \phi_{xy} + (\overline{q}_y)_j \phi_{yy},
\]

\[
p_j(x, y) = \overline{p}_j + (\overline{p}_x)_j \phi_x + (\overline{p}_y)_j \phi_y,
\]

\[
q_j(x, y) = \overline{q}_j + (\overline{q}_x)_j \phi_x + (\overline{q}_y)_j \phi_y.
\]

This is a two-dimensional version of Hyperbolic-RDG-L: 3rd-order advective term and 2nd-order diffusive term. In the case of Hyperbolic-RDG-Q, we have

\[
u_j(x, y) = \overline{u}_j + \overline{p}_j \phi_x + \overline{q}_j \phi_y + (\overline{p}_x)_j \phi_{xx} + (\overline{p}_y)_j \phi_{xy} + (\overline{q}_y)_j \phi_{yy} + \overline{p}_{xx} \phi_{xxx} + \overline{p}_{xy} \phi_{xxy} + \overline{p}_{yy} \phi_{xyy},
\]

\[
p_j(x, y) = \overline{p}_j + (\overline{p}_x)_j \phi_x + (\overline{p}_y)_j \phi_y + (\overline{p}_{xx})_j \phi_{xxx} + (\overline{p}_{xy})_j \phi_{xxy} + (\overline{p}_{yy})_j \phi_{xyy},
\]

\[
q_j(x, y) = \overline{q}_j + (\overline{q}_x)_j \phi_x + (\overline{q}_y)_j \phi_y + (\overline{q}_{xx})_j \phi_{xxx} + (\overline{q}_{xy})_j \phi_{xxy} + (\overline{q}_{yy})_j \phi_{xyy}.
\]

which gives 4th-order advective term and 3rd-order diffusive term. Therefore, the hyperbolic-RDG\((P_0)\)-Q method is a \( P_0 P_3 \) method for advection and a \( P_1 P_2 \) method for diffusion. In all case, the total number of degrees of freedom is three, which is the same, again, as that of the RDG method. Similarly, in 3D also, the hyperbolic-RDG method and the RDG method require the same number of degrees of freedom for third-order accuracy in the advective term, at least. See Table 1.

Extending the discussion to fourth-order accuracy, we obtain the results shown in Table 2. As shown in the table, the hyperbolic-RDG requires slightly more degrees of freedom in 2D and 3D. This is due to the fact,
If this strategy works, then we match the degrees of freedom between the RDG and the hyperbolic RDG for freedom denoted by $\xi_j, \eta_j, \eta_j y_j$. For example, the variables $\xi_j, \eta_j, \eta_j y_j$ replace the first-order partial derivatives $\partial_x u_j$ and $\partial_y u_j$ by a common degree of freedom denoted by $\xi_j$, $\eta_j$, $\eta_j y_j$. This is possible to match the number of degrees of freedom between the RDG and the hyperbolic-RDG with a larger number of degrees of freedom replacing $\eta_j y_j$ by $\xi_j$, for example: Instead of

$$u_j(x, y) = \xi_j + \eta_j \phi_x + \eta_j \phi_y + (\xi_j) \phi_{xx} + (\xi_j y_j) \phi_{xy} + (\eta_j y_j) \phi_{yy}$$

$$p_j(x, y) = \eta_j + (\xi_j) \phi_x + (\eta_j y_j) \phi_y + (\xi_j y_j) \phi_{xx} + (\eta_j y_j) \phi_{xy} + (\eta_j y_j) \phi_{yy},$$

$$q_j(x, y) = \eta_j + (\xi_j) \phi_x + (\eta_j y_j) \phi_y + (\xi_j y_j) \phi_{xx} + (\eta_j y_j) \phi_{xy} + (\eta_j y_j) \phi_{yy},$$

where $\phi_{xx}, \phi_{xy}, \phi_{yy}$, and $\phi_{yy}$ are cubic basis functions, we replace $(\xi_j) y_j$ and $(\eta_j) y_j$ by a common degree of freedom denoted by the first-order partial derivatives $\partial_x u_j$ and $\partial_y u_j$ by a common degree of freedom denoted by $\xi_j$. Instead of

$$u_j(x, y) = \xi_j + \eta_j \phi_x + \eta_j \phi_y + (\xi_j) \phi_{xx} + (\xi_j y_j) \phi_{xy} + (\eta_j y_j) \phi_{yy}$$

$$p_j(x, y) = \eta_j + (\xi_j) \phi_x + (\eta_j y_j) \phi_y + (\xi_j y_j) \phi_{xx} + (\eta_j y_j) \phi_{xy} + (\eta_j y_j) \phi_{yy},$$

$$q_j(x, y) = \eta_j + (\xi_j) \phi_x + (\eta_j y_j) \phi_y + (\xi_j y_j) \phi_{xx} + (\eta_j y_j) \phi_{xy} + (\eta_j y_j) \phi_{yy},$$

If this strategy works, then we match the degrees of freedom between the RDG and the hyperbolic RDG for fourth-order accuracy in all dimensions.
In the RDG method, the reconstructed polynomials are used only in the flux and source computations. It means that only the evolution equation for \( \pi_j \) is derived from the equation for \( u \). Then, the pseudo-time evolution equation of the common degree of freedom \( \pi_{xy,j} \) needs to be derived either from the equation for \( p \) or the equation for \( q \), or a combination of them. How we can actually do so remains to be determined.

3 Remarks

1. The Hyp-RDG-Q method may encounter issues in practical problems as it requires many neighbors for a quadratic reconstruction. But this is the best method, which potentially outperforms the RDG method as it achieves one order higher order of accuracy for the advective part for the same number of degrees of freedom.

2. It is still not clear how to unify the degrees of freedom corresponding to cross derivatives. What equation needs to be solved? I think this is an interesting point to consider in extending the methodology to arbitrarily high-order accuracy. Of course, we can proceed with a full set of degrees of freedom; we just need to carry more degrees of freedom (e.g., 65 instead of 50 in the case of the 3D compressible Navier-Stokes system).

3. The Hyp-RDG-L method is the one comparable to the DG and RDG methods in terms of accuracy for (almost) the same number of degrees of freedom. The approximation order to the diffusive term is one order lower, but it does produce the gradient to the same order of accuracy because the order of accuracy for the gradient is one order lower in the DG and RDG methods. The difference lies in the order of accuracy for \( u \), which can be the same only for advection dominated cases; otherwise one order lower (but still with the same order of accuracy in the gradients).

4. If we compare the DG and hyperbolic-DG methods, we see that Hyp-DG\((P_1)\) is comparable with DG\((P_2)\). In general, Hyp-DG\((P_n)\) is comparable with DG\((P_{n+1})\).

5. I’m interested in higher-order accuracy, i.e., higher than third-order.