Edge-Based Form of Galerkin Source Term Discretization

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Abstract

We here consider Galerkin discretization of a source term. Löhner mentions in Ref.[1] (in a slightly different context) that it can be implemented in a loop over edges, but details are not shown. I suppose that the following is what he meant.

1 Two Dimensions

Consider a triangulation defined by the set \( I \) of nodes and the set \( T \) of triangles. Given solution values at nodes, we consider discretizing a dual-control-volume integral around node \( i \) of a source term, \( s(x,y) \),

\[
\int_{\text{dual}_i} s(x,y) \, dx \, dy,
\]

by the \( P_1 \) Galerkin method. Let \( s_h \) be a piecewise linear representation of \( s \):

\[
s_h = \sum_{i \in I} s_i \phi_i(x,y),
\]

where \( s_i = s(x_i,y_i) \) and \( \phi_i(x,y) \) denotes a piecewise linear basis function that takes 1 at node \( i \) and 0 at all other nodes. The Galerkin discretization of Equation (1) is

\[
\int_{\text{dual}_i} s(x,y) \, dx \, dy \approx \int_{\Omega} \phi_i \, s_h \, dx \, dy
\]

where \( \Omega \) denotes the entire domain. Due to the compactness of \( \phi_i \), it suffices to perform the integration over the set \( \{ T_i \} \) of triangles that share the node \( i \) (see Figure 1). The result is

\[
\int_{\text{dual}_i} s(x,y) \, dx \, dy \approx \frac{1}{6} \sum_{T \in \{ T_i \}} \left( s_i + \frac{1}{2} s_{i\ell} + \frac{1}{2} s_{ir} \right) V_T,
\]

where \( i_\ell \) and \( i_r \) denote the two nodes of the triangle \( T \) other than \( i \), \( V_T \) denotes the volume of the triangle \( T \). This formula can be written as a sum over the neighbors:

\[
\int_{\text{dual}_i} s(x,y) \, dx \, dy \approx \frac{1}{6} \sum_{k \in \{ K_i \}} \frac{s_i + s_k}{2} (V_L + V_R),
\]

where \( \{ K_i \} \) denotes a set of neighbor nodes of \( i \), and \( V_L \) and \( V_R \) are the volumes of the elements on the left and the right of the edge, respectively (see Figure 2). It can be implemented in an edge-loop provided the edge-based data have access to the volumes of the adjacent elements.

Lumped Formula:

If the right hand side is lumped (i.e., \( s_{i\ell} = s_{ir} = s_i \)), the Galerkin discretization (4) becomes

\[
\int_{\text{dual}_i} s(x,y) \, dx \, dy \approx \frac{1}{3} \left( \sum_{T \in \{ T_i \}} V_T \right) s_i = s_i V_{\text{dual}_i},
\]

which is a typical one-point quadrature.
2 Three Dimensions

For a tetrahedral grid defined by the set of tetrahedra, \( \{T_i\} \), we obtain

\[
\int_{\text{dual}_i} s(x, y, z) \, dx \, dy \, dz \approx \int_{\Omega} \phi_i \, s_i \, dx \, dy \, dz = \sum_{T \in \{T_i\}} \left( \frac{1}{10} s_i + \frac{1}{20} s_{i_1} + \frac{1}{20} s_{i_2} + \frac{1}{20} s_{i_3} \right) V_T,
\]

where \( \phi_i \) is a 3D version of the linear basis function at node \( i \), \( \{T_i\} \) denotes the set of tetrahedra sharing the node \( i \), \( V_T \) denotes the volume of the tetrahedron \( T \), and \( i_1, i_2, \) and \( i_3 \) denote the three nodes of the tetrahedron \( T \) other than \( i \). This formula can be written as a sum over the neighbors:

\[
\int_{\text{dual}_i} s(x, y, z) \, dx \, dy \, dz \approx \frac{1}{20} \sum_{k \in \{K_i\}} \left[ \left( \frac{2}{3} s_i + s_k \right) \sum_{T \in \{T_k\}} V_T \right],
\]

where \( \{K_i\} \) denotes a set of neighbor nodes of \( i \), \( \{T_k\} \) denotes the set of tetrahedra sharing the edge \( \{i, k\} \), and \( V_L \) and \( V_R \) are the volumes of the tetrahedra on the left and the right of the edge, respectively. As in two dimensions, the formula reduces to a typical one-point quadrature if we set \( s_{i_1} = s_{i_2} = s_{i_3} = s_i \).

References