# First-Order Hyperbolic System Based Reconstructed Discontinuous Galerkin Methods for Nonlinear Diffusion Equations on Unstructured Grids 

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#### Abstract

This paper presents high-order reconstructed Discontinuous Galerkin (rDG) methods for nonlinear diffusion equations based on the first-order hyperbolic system (FOHS) formulation. Following the previous efforts, efficient high-order schemes are developed by replacing/unifying the derivatives of the primal variable by the gradient variables and their derivatives. However, an existing FOHS formulation for nonlinear equations leads to algorithmic complications since the gradient variables represent diffusive fluxes, not solution gradients, which would require derivatives of a solution-dependent diffusion coefficient. To avoid such complications, a new formulation based on solution gradients is presented and demonstrated for nonlinear diffusion problems. This formulation allows straightforward construction of the efficient hyperbolic rDG schemes for nonlinear equations, but a complication arises in the construction of an upwind diffusion flux. We address this issue and propose a practical simplification. The new formulation and the simplified flux construction are demonstrated numerically for a set of nonlinear diffusion problems.


## I. Introduction

Motivated by the need for efficient diffusion/viscous algorithms in the context of the discontinuous Galerkin (DG) methods and highly-accurate derivative predictions on unstructured grids, we have developed the reconstructed DG methods based on the first-order hyperbolic system (FOHS) formulation. ${ }^{8,9}$ The FOHS formulation introduces additional variables such as solution gradients in order to form a pseudo-time hyperbolic system, and thus allows a straightforward construction of a diffusion/viscous operator by methods for hyperbolic systems. ${ }^{12}$ The additional variables are then used to replace high-order derivatives in the polynomial of the primary solution variables to minimize the total degrees of freedom in the discretization. The resulting high-order rDG schemes have been shown to achieve higher-order accuracy with a less number of degrees of freedom over conventional DG methods despite the fact that additional equations are introduced in the partial-differential-equation level. ${ }^{8,9}$ Extensions to nonlinear systems are possible but requires a change in the FOHS formulation to simplify the construction of numerical schemes. In Ref., ${ }^{14}$ a new FOHS formulation suitable for nonlinear systems was proposed, where the additional variables are defined as the viscous stresses and heat fluxes for the compressible Navier-Stokes equations. This formulation has been further generalized in Ref. ${ }^{16}$ by including the density gradient, so that the gradients of all primitive variables can be obtained from the additional variables. Later, it was extended to three dimensions as presented in Ref. ${ }^{6,11}$ In Ref., ${ }^{6}$ a new construction of a third-order edge-based scheme (Scheme-IQ) was presented, where the second-derivatives of the primitive variables are obtained by the gradients of the viscous stress and heat

[^0]flux variables. Although feasible, extensions of this procedure to higher-order solution derivatives would be highly complicated because it will involve high-order derivatives of the viscosity. This presents a challenge in extending the hyperbolic rDG method to the compressible Navier-Stokes equations and other complex nonlinear system.

To address this issue, we consider a new formulation for nonlinear equations, where the additional variables are taken as solution gradients. This formulation allows us to obtain high-order derivatives of primary variables from the additional variables and their derivatives in a straightforward manner. However, as we will discuss in details, it introduces nonlinear diffusion coefficients in the fluxes. Therefore, the eigen-structure analysis will involve the differentiation of the nonlinear diffusion coefficient, which would introduce complications in the construction of an upwind diffusion flux. To develop a simple and practical scheme, we propose an approximate construction of the upwind dissipation matrix. As will be illustrated, this simplification is similar to the simplification proposed in Ref. ${ }^{15}$ for an advection-diffusion equation.

The objective of the effort discussed in the present work is to develop high-order hyperbolic rDG methods for solving nonlinear diffusion equations. Different reconstruction scheme, including hybrid least-squares $(\mathrm{LS})^{1}$ and variational reconstruction (VR), ${ }^{5,18}$ has been implemented in the study. By combining FOHS and rDG methods, the presented methods can provide high-order results in both primary variables and the derivatives efficiently. The hyperbolic rDG method is a general framework, including finite-volume methods and the method in Ref. ${ }^{10}$ as special cases. A number of nonlinear diffusion problems are presented, indicating the developed hyperbolic rDG method is a cost-effective high-order scheme, and has the potential to ultimately be applied to nonlinear systems such as the compressible Navier-Stokes equations on fully irregular, adaptive, anisotropic, unstructured grids.

The outline of the rest of this paper is organized as follows. A FOHS formulation for a nonlinear diffusion equation is described in Section II. The rDG methods for solving the hyperbolic diffusion equations are presented in Section III. Numerical flux is described in Sections IV. Several numerical experiments are reported in Section V. Concluding remarks and a plan of future work are given in Section VI.

## II. Nonlinear Diffusion Equation and Hyperbolic Formulations

## A. Nonlinear Diffusion Equation

Consider the following model nonlinear diffusion equation in two dimensions.

$$
\begin{equation*}
0=\frac{\partial}{\partial x}\left(\nu(\varphi) \frac{\partial \varphi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\nu(\varphi) \frac{\partial \varphi}{\partial y}\right)+f(x, y) \tag{1}
\end{equation*}
$$

where $\varphi$ denotes a scalar function that can be referred to as velocity potential, $\nu$ is a positive diffusion coefficient, which depends on the solution $\varphi$, and $f(x, y)$ is a source term.

In the hyperbolic method, we introduce additional variables to form a pseudo-time hyperbolic system. The previous papers considered linear diffusion equations and the additional variables were chosen as

$$
\begin{equation*}
u=\frac{\partial \varphi}{\partial x}, \quad v=\frac{\partial \varphi}{\partial y} \tag{2}
\end{equation*}
$$

in the pseudo steady state. For nonlinear equations, however, this choice has not been employed in the past works since it leaves the nonlinear diffusion coefficient in the fluxes and complicates the eigen-structure analysis. To simplify it, the following choice has been employed for nonlinear equations:

$$
\begin{equation*}
u=\nu \frac{\partial \varphi}{\partial x}, \quad v=\nu \frac{\partial \varphi}{\partial y} . \tag{3}
\end{equation*}
$$

This choice has been successfully used for the compressible Navier-Stokes equations in the edge-based discretization up to third-order accuracy. However, this introduces complications in the hyperbolic rDG method, where $u, v$, and their high-order moments are used to construct a high-order polynomial of $\varphi$; or equivalently the derivatives of $\varphi$ are used to represent the polynomials of $u$ and $v$. As we will discuss below, the high-order derivatives of $u$ and $v$ will involve high-order derivatives of $\nu$ for a nonlinear coefficient, and thus obtaining high-order derivatives of $\varphi$ from those of $u$ and $v$ will be a very complicated procedure. In the next section, we begin by considering the latter to further illustrate the point.

## B. FOHS Formulations for Nonlinear Diffusion

## 1. Formulation I

A hyperbolic formulation suitable for nonlinear equations has been proposed and demonstrated for the twodimensional compressible Navier-Stokes equations in Ref., ${ }^{14}$ and later extended to three dimensions. ${ }^{11}$ A hyperbolic viscous system is constructed by introducing the viscous stresses and heat fluxes as additional variables. For the nonlinear diffusion equation (1), it corresponds to the following system:

$$
\left\{\begin{array}{l}
\frac{\partial \varphi}{\partial \tau}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+f(x, y)  \tag{4}\\
\frac{\partial u}{\partial \tau}=\frac{\nu}{T_{r}}\left(\frac{\partial \varphi}{\partial x}-\frac{u}{\nu}\right) \\
\frac{\partial v}{\partial \tau}=\frac{\nu}{T_{r}}\left(\frac{\partial \varphi}{\partial y}-\frac{v}{\nu}\right)
\end{array}\right.
$$

where the additional variable $u$ and $v$ is chosen as the diffusive fluxes, i.e.,

$$
\begin{equation*}
u=\nu \frac{\partial \varphi}{\partial x}, \quad v=\nu \frac{\partial \varphi}{\partial y} \tag{5}
\end{equation*}
$$

and $T_{r}$ is a relaxation time defined by

$$
\begin{equation*}
T_{r}=\frac{L_{r}^{2}}{\nu}, \quad L_{r}=\frac{1}{2 \pi} . \tag{6}
\end{equation*}
$$

Note that $T_{r}$ is a function of $u$ since $\nu=\nu(\varphi)$.
In this formulation, the gradient variables are equivalent to the diffusive fluxes in the pseudo steady state. The gradient of $\varphi$ can be easily obtained as

$$
\begin{equation*}
\frac{\partial \varphi}{\partial x}=\frac{u}{\nu}, \quad \frac{\partial \varphi}{\partial y}=\frac{v}{\nu} \tag{7}
\end{equation*}
$$

However, to obtain higher-order derivatives of the solution $\varphi$ from $(u, v)$, we would need to differentiate the diffusion coefficient. For example, $\partial_{x x} \varphi$ can be obtained as follows:

$$
\begin{equation*}
\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left(\nu \frac{\partial \varphi}{\partial x}\right)=\frac{\partial \nu}{\partial x} \frac{\partial \varphi}{\partial x}+\nu \frac{\partial^{2} \varphi}{\partial x^{2}} \tag{8}
\end{equation*}
$$

and thus

$$
\begin{equation*}
\frac{\partial^{2} \varphi}{\partial x^{2}}=\frac{1}{\nu}\left[\frac{\partial u}{\partial x}-\frac{\partial \nu}{\partial x} \frac{\partial \varphi}{\partial x}\right]=\frac{1}{\nu}\left[\frac{\partial u}{\partial x}-u \frac{\partial \nu}{\partial x}\right] \tag{9}
\end{equation*}
$$

and $\partial_{x x x} \varphi$ is given by

$$
\begin{equation*}
\frac{\partial^{3} \varphi}{\partial x^{3}}=-\frac{1}{\nu^{2}} \frac{\partial \nu}{\partial x}\left[\frac{\partial u}{\partial x}-u \frac{\partial \nu}{\partial x}\right]+\frac{1}{\nu}\left[\frac{\partial^{2} u}{\partial x^{2}}-\frac{\partial u}{\partial x} \frac{\partial \nu}{\partial x}-u \frac{\partial^{2} \nu}{\partial x^{2}}\right] \tag{10}
\end{equation*}
$$

As one can easily imagine, it gets more and more complicated for higher-order derivatives. In the next section, we consider an alternative hyperbolic formulation, which simplifies the process.

## 2. Formulation II

To simplify the process of obtaining higher-order derivatives, we consider the choice:

$$
\begin{equation*}
u=\frac{\partial \varphi}{\partial x}, \quad v=\frac{\partial \varphi}{\partial y} \tag{11}
\end{equation*}
$$

In this case, the gradient of $\varphi$ is directly obtained as $(u, v)$, and thus higher-order derivatives of $\varphi$ can be easily obtained as

$$
\begin{equation*}
\frac{\partial^{2} \varphi}{\partial x^{2}}=\frac{\partial u}{\partial x}, \quad \frac{\partial^{2} \varphi}{\partial y^{2}}=\frac{\partial v}{\partial y}, \quad \frac{\partial^{2} \varphi}{\partial x \partial y}=\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x} \tag{12}
\end{equation*}
$$

$$
\begin{equation*}
\frac{\partial^{3} \varphi}{\partial x^{3}}=\frac{\partial^{2} u}{\partial x^{2}}, \quad \frac{\partial^{3} \varphi}{\partial y^{3}}=\frac{\partial^{2} v}{\partial y^{2}}, \quad \frac{\partial^{3} \varphi}{\partial x^{2} \partial y}=\frac{\partial^{2} u}{\partial x \partial y}=\frac{\partial^{2} v}{\partial x^{2}}, \quad \frac{\partial^{3} \varphi}{\partial x \partial y^{2}}=\frac{\partial^{2} u}{\partial y^{2}}=\frac{\partial^{2} v}{\partial x \partial y}, \tag{13}
\end{equation*}
$$

where all terms on the right hand side in each equation are directly available from the polynomials of $u$ and $v$. Therefore, once the polynomials are defined for all variables, high-order moments of $\varphi$ can be readily expressed by those of $u$ and $v$ to reduce the total number of discrete unknowns as demonstrated in the previous studies for the hyperbolic rDG method. However, a complication arises in the evaluation of the flux Jacobian as discussed below.

This choice (11) leads to the following hyperbolic formulation:

$$
\left\{\begin{array}{l}
\frac{\partial \varphi}{\partial \tau}=\frac{\partial(\nu u)}{\partial x}+\frac{\partial(\nu v)}{\partial y}+f(x, y)  \tag{14}\\
\frac{\partial u}{\partial \tau}=\frac{1}{T_{r}}\left(\frac{\partial \varphi}{\partial x}-u\right) \\
\frac{\partial v}{\partial \tau}=\frac{1}{T_{r}}\left(\frac{\partial \varphi}{\partial y}-v\right)
\end{array}\right.
$$

where $T_{r}$ and $L_{r}$ are still given by Equation (6). In the vector form, we can write

$$
\begin{equation*}
\mathbf{P}^{-1} \frac{\partial \mathbf{U}}{\partial \tau}+\frac{\partial \mathbf{F}_{x}}{\partial x}+\frac{\partial \mathbf{F}_{y}}{\partial y}=\mathbf{S} \tag{15}
\end{equation*}
$$

where

$$
\mathbf{P}^{-1}=\left[\begin{array}{ccc}
1 & 0 & 0  \tag{16}\\
0 & T_{r} & 0 \\
0 & 0 & T_{r}
\end{array}\right], \quad \mathbf{U}=\left[\begin{array}{c}
\varphi \\
u \\
v
\end{array}\right], \quad \mathbf{F}=\left[\begin{array}{c}
-\nu u \\
-\varphi \\
0
\end{array}\right], \quad \mathbf{G}=\left[\begin{array}{c}
-\nu v \\
0 \\
-\varphi
\end{array}\right], \quad \mathbf{S}=\left[\begin{array}{c}
0 \\
-u \\
-v
\end{array}\right] .
$$

The flux projected along an arbitrary vector $\mathbf{n}=\left(n_{x}, n_{y}\right)$ is given by

$$
\mathbf{F}_{n}=\mathbf{F}_{x} n_{x}+\mathbf{F}_{y} n_{y}=\left[\begin{array}{c}
-\nu\left(u n_{x}+v n_{y}\right)  \tag{17}\\
-\varphi n_{x} \\
-\varphi n_{y}
\end{array}\right]
$$

Note that the flux now has the diffusion coefficient, and therefore $\nu$ needs to be differentiated to obtain the flux Jacobian. The preconditioned flux Jacobian is then given by

$$
\mathbf{P A}_{n}=\mathbf{P} \frac{\partial \mathbf{F}_{n}}{\partial \mathbf{U}}=\left[\begin{array}{ccc}
-(\partial \nu / \partial \varphi)\left(u n_{x}+v n_{y}\right) & -\nu n_{x} & -\nu n_{y}  \tag{18}\\
-n_{x} / T_{r} & 0 & 0 \\
-n_{y} / T_{r} & 0 & 0
\end{array}\right]
$$

It is interesting to note that this matrix is very similar to the one for the hyperbolic advection-diffusion system. ${ }^{13}$ It becomes the advection-diffusion Jacobian if we replace $-(\partial \nu / \partial \varphi)\left(u n_{x}+v n_{y}\right)$ by $a n_{x}+b n_{y}$, where $(a, b)$ is the advective vector. Eigenvalues and eigenvectors can be obtained and the dissipation matrix can be constructed as in Ref., ${ }^{13}$ but extensions to more complex systems such as the Navier-Stokes equations would be difficult.

Following the strategy suggested in Refs., ${ }^{14,15}$ where inviscid and viscous Jacobians are treated separately, we propose to split the preconditioned Jacobian into two parts:

$$
\mathbf{P A}_{n}=\mathbf{P A}_{n}^{L}+\mathbf{P A}_{n}^{N}=\left[\begin{array}{ccc}
0 & -\nu n_{x} & -\nu n_{y}  \tag{19}\\
-n_{x} / T_{r} & 0 & 0 \\
-n_{y} / T_{r} & 0 & 0
\end{array}\right]+\left[\begin{array}{ccc}
-(\partial \nu / \partial \varphi)\left(u n_{x}+v n_{y}\right) & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right]
$$

The first term $\mathbf{P A}_{n}^{L}$ corresponds to the linearized-diffusion Jacobian, and the second term $\mathbf{P} A_{n}^{N}$ accounts for the effect of the nonlinear diffusion coefficient. Likewise, the dissipation matrix can be constructed separately for $\mathbf{P A}_{n}^{L}$ and $\mathbf{P A}_{n}^{N}$, and added together to yield

$$
\left|\mathbf{P A}_{n}\right| \approx\left|\mathbf{P A}_{n}^{L}\right|+\left|\mathbf{P A}_{n}^{N}\right|=\sqrt{\frac{\nu}{T_{r}}}\left[\begin{array}{ccc}
1 & 0 & 0  \tag{20}\\
0 & n_{x}^{2} & n_{x} n_{y} \\
0 & n_{x} n_{y} & n_{y}^{2}
\end{array}\right]+\left[\begin{array}{ccc}
\left|(\partial \nu / \partial u)\left(u n_{x}+v n_{y}\right)\right| & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right]
$$

The second term is considered as a nonlinear correction.
In this paper, Formulation II is explored for nonlinear diffusion equations, and demonstrated with the above simplified approach for the dissipation matrix construction.

## III. Reconstruction Discontinuous Galerkin Methods

In this section, we describe the rDG discretization of the hyperbolic formulation. We assume that the domain $\Omega$ is subdivided into a collection of non-overlapping arbitrary elements $\Omega_{e}$, and then introduce the following broken Sobolev space $V_{h}^{n}$

$$
\begin{equation*}
V_{h}^{n}=\left\{v_{h} \in\left[L^{2}(\Omega)\right]^{k}:\left.v_{h}\right|_{\Omega_{e}} \in\left[V_{n}^{k}\right] \forall \Omega_{e} \in \Omega\right\} \tag{21}
\end{equation*}
$$

which consists of discontinuous vector polynomial functions of degree $n$, and where $k$ is the dimension of the unknown vector and $V_{n}$ is the space of all polynomials of degree $\leq n$. To formulate the discontinuous Galerkin method, we introduce the following weak formulation, which is obtained by multiplying Eq. (14) by a test function $\mathbf{W}_{h}$, integrating over an element $\Omega_{e}$, and then performing an integration by parts: find $\mathbf{U}_{h} \in V_{h}^{p}$ such as

$$
\begin{equation*}
\frac{d}{d \tau} \int_{\Omega_{e}} \mathbf{W}_{h} \overline{\mathbf{P}}_{e}^{-1} \mathbf{U}_{h} d \Omega+\int_{\Gamma_{e}} \mathbf{W}_{h} \mathbf{F}_{k} \mathbf{n}_{k} d \Gamma-\int_{\Omega_{e}} \frac{\partial \mathbf{W}_{h}}{\partial x_{k}} \mathbf{F}_{k} d \Omega=\int_{\Omega_{e}} \mathbf{W}_{h} \mathbf{S} d \Omega, \quad \forall \mathbf{W}_{h} \in V_{h}^{n} \tag{22}
\end{equation*}
$$

where $\mathbf{F}=\left[\mathbf{F}_{x}, \mathbf{F}_{y}\right], \mathbf{U}_{h}$ and $\mathbf{W}_{h}$ are represented by piecewise polynomial functions of degrees $p$, which are discontinuous between the cell interfaces, and $\mathbf{n}_{k}$ the unit outward normal vector to the $\Gamma_{e}$ : the boundary of $\Omega_{e}$. Note in particular that the matrix $\overline{\mathbf{P}}_{e}^{-1}$ has been evaluated at a local value with the cell-average solution and taken as constant within an element. Besides, with the dimensionless modal DG formulation, the cell-average solution is handily available, which make the construction of the preconditioning matrix $\mathbf{P}$ very straightforward. This simplification does not have any impact on accuracy because the solution is sought in the pseudo steady state.

The standard DG solution $\mathbf{U}_{h}$ within the element $\Omega_{e}$ can be expressed as

$$
\begin{equation*}
\mathbf{U}_{h}(x, y, \tau)=\mathbf{C}(x, y) \mathbf{V}(\tau) \tag{23}
\end{equation*}
$$

where $\mathbf{C}$ is a basis matrix, and $\mathbf{V}$ is a vector of unknown polynomial coefficients. If we set the test function $\mathbf{W}_{h}$ as the transpose of the basis matrix $\mathbf{C}$, then the following equivalent system would be arrived.

$$
\begin{equation*}
\frac{d}{d \tau} \int_{\Omega_{e}} \mathbf{C}^{T} \overline{\mathbf{P}}_{e}^{-1} \mathbf{C V} d \Omega+\int_{\Gamma_{e}} \mathbf{C}^{T} \mathbf{F}_{k} \mathbf{n}_{k} d \Gamma-\int_{\Omega_{e}} \frac{\partial \mathbf{C}^{T}}{\partial x_{k}} \mathbf{F}_{k} d \Omega=\int_{\Omega_{e}} \mathbf{C}^{T} \mathbf{S} d \Omega \tag{24}
\end{equation*}
$$

This scheme is called discontinuous Galerkin method of degree $n$, or in short notation $\mathrm{DG}\left(\mathrm{P}_{n}\right)$ method. By simply increasing the degree $n$ of the polynomials, the DG methods of corresponding higher order are obtained. In the rDG method, inspired by methods from Dumbser et al. $\mathrm{P}_{n} \mathrm{P}_{m}$ scheme, ${ }^{2-4}$ a higher-order solution is reconstructed from the underlying solution polynomials. For rDG $\left(\mathrm{P}_{n} \mathrm{P}_{m}\right)$ method with $m>n$, a higher-order reconstructed numerical solution can be obtained:

$$
\begin{equation*}
\mathbf{U}_{h}^{R}(x, y, \tau)=\mathbf{C}^{R}(x, y) \mathbf{V}^{R}(\tau) \tag{25}
\end{equation*}
$$

where higher-order derivatives (higher than $n$-th and up to $m$-th) are reconstructed from the underlying $\mathrm{P}_{n}$ solution. There are three approaches to the reconstruction. One is a least-squares reconstruction method,
and another is a variational reconstruction method. The last option, which is unique in the FOHS formulation considered here, is to directly use the gradient variables and their moments to evaluate these derivatives. Or equivalently, this approach can be thought of as defining the solution as $\mathrm{P}_{m}$, and use higher-order moments to represent the gradient variables in the FOHS formulation. In the former two approaches, the method is expressed by $\operatorname{rDG}\left(\mathrm{P}_{n} \mathrm{P}_{m}\right)$, and the latter approach by $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{m}\right)$ since high-order derivatives are already available and no explicit reconstruction is required. This higher order numerical solution $\mathbf{U}_{h}^{R}$ would be used for flux and source term computation.

By moving the second and third terms to the right-hand-side (r.h.s.) in Eq. (24), we will arrive at a system of ordinary differential equations (ODEs) in time, which can be written in semi-discrete form as

$$
\begin{equation*}
\mathbf{M} \frac{d \mathbf{V}}{d t}=\mathbf{R}\left(\mathbf{U}_{h}^{R}\right) \tag{26}
\end{equation*}
$$

where $\mathbf{M}$ is the mass matrix,

$$
\begin{equation*}
\mathbf{M}=\int_{\Omega_{e}} \mathbf{C}^{T} \overline{\mathbf{P}}_{e}^{-1} \mathbf{C} d \Omega \tag{27}
\end{equation*}
$$

and $\mathbf{R}$ is the residual vector, defined as

$$
\begin{equation*}
\mathbf{R}=\left[\int_{\Omega_{e}} \frac{\partial \mathbf{C}^{T}}{\partial x_{k}} \mathbf{F}_{k}\left(\mathbf{U}_{h}^{R}\right)+\mathbf{C}^{T} \mathbf{S}\left(\mathbf{U}_{h}^{R}\right) d \Omega-\int_{\Gamma_{e}} \mathbf{C}^{T} \mathbf{F}_{k}\left(\mathbf{U}_{h}^{R}\right) \mathbf{n}_{k} d \Gamma\right] \tag{28}
\end{equation*}
$$

In this study,GMRES+LU-SGS and GCR+SGS(k) have been developed to solve the linear system, where LU-SGS/SGS(k) serve as the preconditioner, where $k$ is the number of relaxations.

Based on different rDG methods, some effective discretization hyperbolic rDG methods will be presented to deal with the derived FOHS. The format $\mathbf{A}+\mathbf{B}$ is used to indicate the discretization method for the system, where $\mathbf{A}$ refers to the discretization method for $\varphi$ and $\mathbf{B}$ refers to the discretization method for its derivatives. Different choices and combinations for $\mathbf{A}$ and $\mathbf{B}$ are compared in the authors' previous work. ${ }^{7}$ To minimize the memory and storage cost of the developed methods, one can apply DG $\left(\mathrm{P}_{n}\right)$ or rDG $\left(\mathrm{P}_{n} \mathrm{P}_{m}\right)$ methods only on the derivative variables. With the handily information of the derivatives, a higher order of polynomial for $\varphi$ can be constructed with only one degree of freedom. Therefore, in this paper, we would focus on $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{n+1}\right)+\mathrm{DG}\left(\mathrm{P}_{n}\right)$ and $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{m+1}\right)+\mathrm{rDG}\left(\mathrm{P}_{n} \mathrm{P}_{m}\right)$ methods. Further details can be found in the previous papers.

## IV. Numerical Flux

For the preconditioned system, the numerical flux is constructed as

$$
\begin{equation*}
\mathbf{F}_{i j}\left(\mathbf{U}_{L}, \mathbf{U}_{R}\right)=\frac{1}{2}\left[\mathbf{F}_{n}\left(\mathbf{U}_{L}\right)+\mathbf{F}_{n}\left(\mathbf{U}_{R}\right)\right]-\frac{1}{2} \mathbf{P}^{-1}\left|\mathbf{P} A_{n}\right|\left(\mathbf{U}_{R}-\mathbf{U}_{L}\right) \tag{29}
\end{equation*}
$$

where $\mathbf{A}_{n}$ is the flux Jacobian. The construction of the dissipation matrix follows a standard technique in the local-preconditioning method. ${ }^{17}$ As discusses earlier, we employ a simplified approach and construct the absolute Jacobian as

$$
\begin{equation*}
\mathbf{F}_{i j}\left(\mathbf{U}_{L}, \mathbf{U}_{R}\right)=\frac{1}{2}\left[\mathbf{F}_{n}\left(\mathbf{U}_{L}\right)+\mathbf{F}_{n}\left(\mathbf{U}_{R}\right)\right]-\frac{1}{2} \mathbf{P}^{-1}\left(\left|\mathbf{P A}_{n}^{L}\right|+\left|\mathbf{P} A_{n}^{R}\right|\right)\left(\mathbf{U}_{R}-\mathbf{U}_{L}\right), \tag{30}
\end{equation*}
$$

where $\left|\mathbf{P A}_{n}^{L}\right|+\left|\mathbf{P A}_{n}^{R}\right|$ is given by Equation (20). The dissipation matrix depends on the solution for nonlinear equations, and it is evaluated by the arithmetic average: $\left(\mathbf{U}_{L}+\mathbf{U}_{R}\right) / 2$.

## V. Numerical Results

A steady model nonlinear diffusion problem in a unit square is considered in this section, i.e.,

$$
\begin{equation*}
\frac{\partial \varphi}{\partial t}=\frac{\partial}{\partial x}\left(\nu(\varphi) \frac{\partial \varphi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\nu(\varphi) \frac{\partial \varphi}{\partial y}\right)+f(x, y) \tag{31}
\end{equation*}
$$

with the exact solution, the diffusion coefficient and the source term given by

$$
\begin{equation*}
\varphi(x, y)=\sin (\pi x) \sin (\pi y) \tag{32}
\end{equation*}
$$

$$
\begin{equation*}
\nu=\nu(\varphi)=\varphi^{2}+1, \tag{33}
\end{equation*}
$$

and

$$
\begin{equation*}
f(x, y)=2 \pi^{2} \varphi\left(3 \cos ^{2}(\pi x) \cos ^{2}(\pi y)-2 \cos ^{2}(\pi x)-2 \cos ^{2}(\pi y)+2\right) \tag{34}
\end{equation*}
$$

In this paper, three sets of meshes are used in the test, namely regular, irregular and heterogeneous grids. The sample of each type of grids are shown in Figure 1.

The grid refinement study has been carried out using the developed hyperbolic rDG methods. The results are shown for each type of mesh in Table 1 and Figure 2 to 4.


Figure 1: The sample mesh of each type, i.e., $17 \times 17$ regular grid (left), $17 \times 17$ irregular grid (middle), and $23 \times 21$ heterogeneous grid (right).

Table 1: Order of accuracy on different type of grids.

|  | Regular grids |  | Irregular grids |  | Heterogeneous grids |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\varphi$ | $\mathrm{v}_{x}$ | $\varphi$ | $\mathrm{v}_{x}$ | $\varphi$ | $\mathrm{v}_{x}$ |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{1}\right)+\mathrm{DG}\left(\mathrm{P}_{0}\right)$ | 0.93 | 1.00 | 0.93 | 0.97 | 1.00 | 1.00 |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{2}\right)+\mathrm{DG}\left(\mathrm{P}_{1}\right)$ | 1.93 | 1.88 | 1.80 | 1.56 | 1.82 | 1.86 |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{3}\right)+\mathrm{DG}\left(\mathrm{P}_{2}\right)$ | 3.76 | 2.73 | 3.84 | 2.77 | 3.82 | 2.72 |
| DG( $\left.\mathrm{P}_{0} \mathrm{P}_{2}\right)+$ rDG_LS $\left(\mathrm{P}_{0} \mathrm{P}_{1}\right)$ | 2.28 | 2.07 | 2.14 | 2.02 | 1.93 | 1.91 |
| DG $\left(\mathrm{P}_{0} \mathrm{P}_{3}\right)+$ rDG_LS $\left(\mathrm{P}_{1} \mathrm{P}_{2}\right)$ | 3.83 | 3.02 | 3.84 | 2.89 | 3.40 | 2.77 |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{2}\right)+\mathrm{rDG}-\mathrm{VR}\left(\mathrm{P}_{0} \mathrm{P}_{1}\right)$ | 1.97 | 1.99 | 1.96 | 1.91 | 1.82 | 1.89 |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{3}\right)+\mathrm{rDG}$-VR $\left(\mathrm{P}_{0} \mathrm{P}_{2}\right)$ | 3.94 | 3.35 | 4.02 | 3.47 | 3.34 | 2.60 |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{3}\right)+\mathrm{rDG}-\mathrm{VR}\left(\mathrm{P}_{1} \mathrm{P}_{2}\right)$ | 3.83 | 3.02 | 3.76 | 3.00 | 3.53 | 2.41 |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{4}\right)+\mathrm{rDG}$-VR $\left(\mathrm{P}_{1} \mathrm{P}_{3}\right)$ | 4.02 | 3.85 | 4.01 | 3.79 | 3.96 | 4.33 |



Figure 2: Grid refinement study on regular grids.


Figure 3: Grid refinement study on irregular grids.


Figure 4: Grid refinement study on heterogeneous grids.

Overall, the hyperbolic rDG methods delivered the designed order of accuracy for the case shown above. Note that several presented schemes like $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{3}\right)+\mathrm{rDG} \mathrm{LS}^{2}\left(\mathrm{P}_{1} \mathrm{P}_{2}\right)$, $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{3}\right)+\mathrm{rDG} \mathrm{DR}^{2}\left(\mathrm{P}_{0} \mathrm{P}_{2}\right)$, and $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{3}\right)+\mathrm{rDG}-\mathrm{VR}\left(\mathrm{P}_{1} \mathrm{P}_{2}\right)$ are able to deliver 4 th order in $\varphi$ and 3rd order in gradients in all the grids very effectively. Note that these results are better than expected since the hyperbolic scheme typically achieves the same order of accuracy (based on the lowest order of polynomials) for all variables, e.g., $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{m+1}\right)+\mathrm{rDG}\left(\mathrm{P}_{0} \mathrm{P}_{m}\right)$ is expected to yield $(m+1)$-th order of accuracy for all variables. As for $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{4}\right)+\mathrm{rDG}-\mathrm{VR}\left(\mathrm{P}_{1} \mathrm{P}_{3}\right)$ can obtain fourth order of accurate gradients. Additionally, one can find that variational reconstruction based rDG schemes are more stable than the least-squares rDG counterpart. Both $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{2}\right)+\mathrm{rDG}$ LS $\left(\mathrm{P}_{0} \mathrm{P}_{1}\right)$ and $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{3}\right)+\mathrm{rDG}$ LS $\left(\mathrm{P}_{1} \mathrm{P}_{2}\right)$ are unable to deliver stable results for the finest heterogeneous grids without any limiter. On the contrary, the counterparts with variational reconstruction are stable and can deliver the desired order of accuracy for all the grids. Meanwhile, for variational reconstruction, one can have global stencil with compact data structure, thus to resolve the stability issue and make the extension to higher order reconstruction more straightforward. Also, boundary condition can be ignored for using variational reconstruction. The numerical results indicate that the presented hyperbolic rDG schemes are attractive and worth further investigation.

## VI. Conclusions and Outlook

High order reconstructed discontinuous Galerkin (rDG) methods based on first-order hyperbolic system (FOHS) for nonlinear diffusion equations have been developed and presented in the study. With FOHS formulation, an equivalent hyperbolic system, which would yield at the same steady solution, is generated. Instead of using the diffusive fluxes as the additional variables in the FOHS, the developed new formulation adopts the gradients of the primary variables as the auxiliary variables, leading to a straightforward approach for obtaining arbitrary high-order moments of the primary variable. The numerical examples showed in the paper illustrate the capability and the potential of the developed methods, indicating that the hyperbolic rDG methods provide attractive alternatives to solve nonlinear diffusion equations. A ongoing effort is being taken for the time-dependent problems. Future work would also be focused on extending the hyperbolic rDG method to Navier-Stokes equation on fully 3D unstructured grids.

## Acknowledgments

This work has been funded by the U.S. Army Research Office under the contract/grant number W911NF-16-1-0108 with Dr. Matthew Munson as the program manager.

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