AIAA AVIATION Forum
June 25-29, 2018, Atlanta, Georgia
2018 Fluid Dynamics Conference

10.2514/6.2018-4270

Explicit Hyperbolic Reconstructed Discontinuous Galerkin Methods for Time-Dependent Problems

Jialin Lou *, Lingquan Li † and Hong Luo ‡
North Carolina State University, Raleigh, NC 27695, United States
and
Hiroaki Nishikawa §
National Institute of Aerospace, Hampton, VA 23666, United States

This paper reports a new finding that time-accurate explicit time-stepping schemes can be constructed by the reconstructed discontinuous Galerkin method applied to a hyperbolic diffusion formulation that is consistent with a diffusion equation only in the steady state. A naive discretization of the hyperbolic diffusion system is known to be time inconsistent, but a special Galerkin discretization with a matrix basis used as a test function is found to yield a time-accurate semi-discrete system that can be integrated in time by explicit time-stepping schemes. The mechanism behind the unexpected property is discussed. The presented schemes are more efficient than conventional discontinuous Galerkin schemes with higher-order accuracy achieved with a fewer degrees of freedom. Two reconstruction techniques are considered for the efficient construction: least-squares reconstruction and variational reconstruction. Unsteady computations for pure diffusion problems and advection-diffusion problems are presented to assess accuracy and performance of the newly developed high-order explicit hyperbolic reconstructed discontinuous Galerkin schemes. Numerical experiments demonstrate that the explicit hyperbolic reconstructed discontinuous Galerkin schemes achieve the designed optimal order of accuracy for both solutions and their derivatives on regular and irregular grids for unsteady problems.

I. Introduction

This paper presents explicit time-accurate high-order hyperbolic reconstructed discontinuous Galerkin (rDG) schemes. The hyperbolic rDG method is a special discretization method that combines the hyperbolic diffusion formulation [1] and the rDG discretization methods [2–7]. The hyperbolic diffusion formulation is a first-order system form of a diffusion equation with extra variables, called the gradient variables, added to form a system and with pseudo time terms added to render the system hyperbolic. The rDG method is a general framework for constructing efficient high-order schemes with reconstruction techniques, having the finite-volume (FV) and DG schemes as special cases. As we have shown in our previous developments [8–12], the two approaches can be combined in a systematic manner to simplify the discretization of diffusion terms, improve gradient accuracy, accelerate iterative convergence, and achieve higher-order accuracy than DG methods with fewer numbers of degrees of freedom. Specifically, we have developed hyperbolic rDG schemes for diffusion with scalar and tensor diffusion coefficients [8, 9], nonlinear diffusion [10] advection-diffusion equations [11], and the Navier-Stokes equations [12]. For unsteady problems, we followed Refs.[13, 14], and employed implicit time-stepping schemes. The major reason for implicit-time stepping lies in the fact that the hyperbolic diffusion formulations rely on the equivalent to the original diffusion/viscous equations in the pseudo steady state, and therefore are not necessarily time-accurate when the pseudo time is treated as the physical time. However, as a matter of fact, the hyperbolic method is a method for spatial discretizations:

*Ph.D Student, Department of Mechanical and Aerospace Engineering, Student Member AIAA.
†Ph.D Student, Department of Mechanical and Aerospace Engineering, Student Member AIAA.
‡Professor, Department of Mechanical and Aerospace Engineering, Associate Fellow AIAA.
§Associate Research Fellow, National Institute of Aerospace, Associate Fellow AIAA.

form a first-order diffusion system with a pseudo time term such that the system is hyperbolic in the pseudo time, discretize it by upwind schemes, and drop the pseudo time term to obtain a spatial discretization. Seemingly, then, it can be combined with any time-stepping scheme. However, as we will discuss later, the resulting schemes would lose the coupling among the variables in the discrete equations if explicit time-stepping schemes are used. In effect, such schemes reduce to conventional diffusion schemes, and lose all the benefits of the hyperbolic method. To keep the coupling in the discrete level, implicit-time stepping schemes needed to be employed, where a steady solver is used for solving fully coupled unsteady residual equations. In this paper, we report a quite remarkable finding that time-accurate explicit time-stepping schemes arise automatically from the rDG or DG method applied to the hyperbolic diffusion formulation.

The hyperbolic diffusion formulation [1] is similar to the classical hyperbolic heat equations of Cattaneo [15] and Vernotte [16], with a relaxation time parameter, T_r. These classical equations establish the equivalence to the original second-order diffusion equation in the limit $T_r \to 0$, thus resulting in a system with stiff source terms. On the other hand, the hyperbolic method considered here is different from these classical models in that the hyperbolic formulation is constructed with a pseudo time term and designed to be equivalent to the diffusion term in the pseudo steady state for arbitrary relaxation time. The relaxation time can be taken as $O(1)$, designed to optimize iterative convergence [1], and thus there are no stiff source terms. Because of the large relaxation time, the hyperbolic formulation will not be time-accurate when the pseudo-time term treated as the physical-time term. It is possible to recover time accuracy by taking a very small relaxation time, and it is useful to be able to employ explicit time-stepping schemes, which are simpler and computationally inexpensive compared with implicit counterparts, especially when problems are not stiff, e.g., no boundary layers are involved. In this regard, Toro and Montecinos [17] analyzed a similar hyperbolic diffusion system and derived an upper bound on T_r, below which time accuracy can be obtained with explicit schemes. Specifically, they derived the upper bound as $T_r = O(h^{1+r/2})$, where h is a mesh spacing and r is the accuracy order (e.g., $r = 2$ for second-order accurate schemes), and demonstrated its validity for schemes up to seventh-order accuracy.

In this paper, we show that explicit time-accurate hyperbolic schemes can be constructed without any careful adjustment to the relaxation time. Rather surprisingly, we have found that the rDG or DG discretization applied to the hyperbolic diffusion formulation is automatically time-accurate with explicit time-stepping, with arbitrary relaxation time of $O(1)$. As will be shown, the rDG/DG formulation automatically generates appropriate time derivatives for the gradient variables, and creates a semi-discrete system that can be accurately integrated in time with explicit schemes. It is also demonstrated that the resulting explicit hyperbolic rDG schemes can yield the same order of accuracy for the primal solution and its gradients on irregular grids. This is one of the unique features of the hyperbolic method, as demonstrated in many previous papers, and would be highly desired for unstructured-grid applications with both explicit and implicit time-stepping schemes. For reconstruction, both hybrid least-squares (LS) [18] and variational reconstruction (VR) [19] are considered. As demonstrated in Ref.[11], the developed schemes are more efficient than the underlying DG schemes. Unsteady computations are presented for pure diffusion problems and advection-diffusion equations. Numerical results demonstrate that the hyperbolic rDG method is a cost-effective high-order method, and encourage applications to the incompressible and compressible Navier-Stokes equations.

The paper is organized as follows. The difficulty in the construction of explicit hyperbolic diffusion schemes is illustrated in Section II. The fact that it is not at all difficult in the case of the hyperbolic rDG schemes is discussed in Section III. Numerical experiments are reported in Section IV. Concluding remarks and a plan of future work are given in Section V.

II. Difficulty in Explicit Hyperbolic Diffusion Schemes

Consider the time-dependent diffusion equation

$$\frac{\partial \varphi}{\partial t} = \nu \frac{\partial^2 \varphi}{\partial x^2}, \tag{1}$$

where φ denotes a scalar function that can be referred to as velocity potential, and ν is a constant. A classical hyperbolic diffusion formulation [15, 16] is given by

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} = \nu \dfrac{\partial \mathrm{v}_x}{\partial x}, \\ \dfrac{\partial \mathrm{v}_x}{\partial t} = \dfrac{1}{\epsilon}\left(\dfrac{\partial \varphi}{\partial x} - \mathrm{v}_x \right), \end{cases} \tag{2}$$

where ϵ is a small constant and v_x is an additional variable denoted as the gradient of the primary variable φ. This system is equivalent to the original diffusion equation for a sufficiently small ϵ. Montecinos and Toro [17, 20] derived an upper bound of ϵ, which preserves a design order of accuracy, and developed high-order schemes for the hyperbolic formulation. In this formulation, explicit time integration schemes can be easily employed because the hyperbolic formulation is equivalent to the diffusion equation in the partial-differential-equation level. However, improved accuracy in the solution gradients on irregular grids, as demonstrated in Ref.[21], is not confirmed in this approach.

In the hyperbolic method [1], the same hyperbolic system is employed:

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} = \nu \dfrac{\partial \mathrm{v}_x}{\partial x}, \\ \dfrac{\partial \mathrm{v}_x}{\partial t} = \dfrac{1}{T_r}\left(\dfrac{\partial \varphi}{\partial x} - \mathrm{v}_x \right), \end{cases} \tag{3}$$

but the relaxation time T_r is defined as

$$T_r = \frac{L_r^2}{\nu}, \quad L_r = \frac{1}{2\pi}, \tag{4}$$

which has been derived by requiring Fourier modes to propagate for fast steady convergence [1]. As discussed in Ref.[1], this relaxation time is too large to ensure the consistency with the original diffusion equation, but the equivalence is guaranteed in the steady state. The steady equivalence is the key idea in the hyperbolic method, by which spatial discretizations with special features (e.g., higher-order accuracy in the gradient and convergence acceleration) can be constructed. In the hyperbolic method, time-accurate schemes are constructed based on the following form:

$$\begin{cases} \dfrac{\partial \varphi}{\partial \tau} + \dfrac{\partial \varphi}{\partial t} = \nu \dfrac{\partial \mathrm{v}_x}{\partial x}, \\ \dfrac{\partial \mathrm{v}_x}{\partial \tau} = \dfrac{1}{T_r}\left(\dfrac{\partial \varphi}{\partial x} - \mathrm{v}_x \right), \end{cases} \tag{5}$$

where τ is a pseudo time variable. In the pseudo steady state (or as soon as the pseudo time derivatives are dropped), we recover the equivalence with the original usteady diffusion equation. The system being hyperbolic in τ, we can discretize the spatial part by upwind methods, and drop the pseudo time derivatives to obtain:

$$\begin{cases} \dfrac{d\varphi_h}{dt} = \nu \left(\dfrac{\partial \mathrm{v}_x}{\partial x} \right)_h, \\ 0 = \dfrac{1}{T_r}\left(\dfrac{\partial \varphi}{\partial x} - \mathrm{v}_x \right)_h, \end{cases} \tag{6}$$

where the subscript h indicates the discrete approximation. Then, time-accurate schemes are constructed by implicit time-integration schemes, e.g., by the backward Euler scheme:

$$\begin{cases} \dfrac{\varphi_h^{n+1} - \varphi_h^n}{\Delta t} = \nu \left(\dfrac{\partial \mathrm{v}_x}{\partial x} \right)_h^{n+1}, \\ 0 = \dfrac{1}{T_r}\left(\dfrac{\partial \varphi}{\partial x} - \mathrm{v}_x \right)_h^{n+1}, \end{cases} \tag{7}$$

where Δt is a time step, and n indicates the time level. These equations are fully coupled and need to be solved simultaneously at every time step to obtain φ_h^{n+1} and $(\mathrm{v}_x)_h^{n+1}$. The variable coupling is essential

to ensuring the properties of hyperbolic schemes: same order of accuracy for φ and v_x, and fast iterative convergence in solving the unsteady residual equations.

On the other hand, if the forward Euler scheme is employed, we have

$$\begin{cases} \dfrac{\varphi_h^{n+1}-\varphi_h^n}{\Delta t} = \nu \left(\dfrac{\partial \mathrm{v}_x}{\partial x}\right)_h^n, \\ 0 = \dfrac{1}{T_r}\left(\dfrac{\partial \varphi}{\partial x} - \mathrm{v}_x\right)_h^n. \end{cases} \tag{8}$$

In this case, the equations are decoupled: φ_h^{n+1} is immediately updated by the first equation alone, and there is no update scheme for the other variable $(\mathrm{v}_x)_h^{n+1}$. In effect, the second equation has become redundant. One may solve $(\partial_x\varphi - \mathrm{v}_x)_h^{n+1} = 0$ for $(\mathrm{v}_x)_h^{n+1}$, but this is simply a gradient reconstruction and therefore the equal order of accuracy for φ and v_x, which is one of the advantages of the hyperbolic method, is likely to be lost on irregular grids. In short, this is just a conventional diffusion scheme with gradient reconstruction.

The above argument explains why implicit time integration schemes have been employed exclusively in the hyperbolic method in all previous papers. Below, it is discussed that explicit time-stepping schemes can be constructed in a rather straightforward manner in the hyperbolic DG method.

III. Explicit Hyperbolic Reconstruction Discontinuous Galerkin Methods

A. DG Discretization

Consider the hyperbolic diffusion system in the vector form:

$$\frac{\partial \mathbf{U}}{\partial \tau} + \mathbf{T}\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}_x}{\partial x} = \mathbf{S}, \tag{9}$$

where

$$\mathbf{U} = \begin{pmatrix} \varphi \\ \mathrm{v}_x \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{F}_x = \begin{pmatrix} -\nu \mathrm{v}_x \\ -\varphi/T_r \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 0 \\ -\mathrm{v}_x/T_r \end{pmatrix}. \tag{10}$$

We begin by formulating the DG method for the hyperbolic diffusion system. We assume that the domain Ω is subdivided into a collection of non-overlapping arbitrary elements Ω_e, and then introduce the following broken Sobolev space V_h^n:

$$V_h^n = \left\{ v_h \in \left[L^2(\Omega)\right]^k : v_h|_{\Omega_e} \in \left[V_n^k\right] \forall \Omega_e \in \Omega \right\}, \tag{11}$$

which consists of discontinuous vector polynomial functions of degree n, and where k is the dimension of the unknown vector and V_n is the space of all polynomials of degree $\leq n$. To formulate the DG method, we introduce the following weak formulation, which is obtained by multiplying Eq. (9) by a test function $\mathbf{W}_h$, integrating over an element Ω_e, and then performing an integration by parts: find $\mathbf{U}_h \in V_h^p$ such as

$$\begin{aligned} &\frac{\partial}{\partial \tau}\int_{\Omega_i} \mathbf{W}_h \mathbf{U}_h d\Omega + \frac{\partial}{\partial t}\int_{\Omega_i} \mathbf{W}_h \mathbf{T}\mathbf{U}_h d\Omega + \int_{\Gamma_i} \mathbf{W}_h \mathbf{F}_k \mathbf{n}_k d\Gamma \\ &- \int_{\Omega_i} \frac{\partial \mathbf{W}_h}{\partial x_k}\mathbf{F}_k d\Omega = \int_{\Omega_i} \mathbf{W}_h \mathbf{S} d\Omega, \qquad \forall \mathbf{W}_h \in V_h^n, \end{aligned} \tag{12}$$

where $\mathbf{U}_h$ and $\mathbf{W}_h$ are represented by piecewise polynomial functions of degrees p, which are discontinuous between the cell interfaces, and $\mathbf{n}_k$ the unit outward normal vector to the Γ_i: the boundary of Ω_i. The standard DG solution $\mathbf{U}_h$ within the element Ω_i can be expressed as

$$\mathbf{U}_h(x,t,\tau) = \mathbf{C}(x)\mathbf{V}(t,\tau), \tag{13}$$

where $\mathbf{C}$ is a basis matrix, and $\mathbf{V}$ is a vector of unknown polynomial coefficients. In the implementation of the DG methods in this paper, modal based DG methods are adopted. The numerical polynomial solutions are represented using a Taylor series expansion at the cell center and normalized to improve the conditioning of the system matrix. For instance, according to the Taylor expansion in 1D, one would have

$$\varphi_h = \overline{\varphi} + \varphi_x^c \Delta x B_2 + \varphi_{xx}^c \Delta x^2 B_3 + \varphi_{xxx}^c \Delta x^3 B_4 + \cdots, \tag{14}$$

where the $\overline{\varphi}$ represents the averaged quantity of φ, and the superscript c stands for the central values. The basis functions are given as follows

$$B_1 = 1, B_2 = \frac{x - x_c}{\Delta x}, B_3 = \frac{1}{2}\left(B_2^2 - \frac{1}{\Omega_e}\int_{\Omega_e} B_2^2 d\Omega\right), B_4 = \frac{1}{6}\left(B_2^3 - \frac{1}{\Omega_e}\int_{\Omega_e} B_2^3 d\Omega\right), \tag{15}$$

where

$$\Delta x = 0.5(x_{\max} - x_{\min}) = 0.5h. \tag{16}$$

A more efficient construction can be devised [11], in which the basis functions form a matrix $\mathbf{C}$. A further discussion about $\mathbf{C}$ and $\mathbf{V}$ will be given later in the next section. If we set the test function $\mathbf{W}_h$ as the transpose of the basis matrix $\mathbf{C}$, then we obtain the following:

$$\frac{\partial}{\partial \tau}\int_{\Omega_i} \mathbf{C}^T\mathbf{C}\mathbf{V} d\Omega + \frac{\partial}{\partial t}\int_{\Omega_i} \mathbf{C}^T\mathbf{T}\mathbf{C}\mathbf{V} d\Omega + \int_{\Gamma_i} \mathbf{C}^T\mathbf{F}_k\mathbf{n}_k d\Gamma - \int_{\Omega_i} \frac{\partial \mathbf{C}^T}{\partial x_k}\mathbf{F}_k d\Omega = \int_{\Omega_i} \mathbf{C}^T\mathbf{S} d\Omega. \tag{17}$$

The integrals are evaluated by Gaussian quadrature rules of appropriate orders, and the flux at the interface will be computed art each quadrature point for a given set of solution values obtained from the polynomials defined on two elements sharing the interface. In this paper, the following upwind flux is employed for the diffusion term [22]:

$$\mathbf{F}_{i+1/2} = \frac{1}{2}(\mathbf{F}_L + \mathbf{F}_R) - \frac{1}{2}\sqrt{\frac{\nu}{T_r}}(\mathbf{U}_R - \mathbf{U}_L), \tag{18}$$

where $i + 1/2$ indicates an interface, and the subscripts L and R indicate the solutions at the interface evaluated by the polynomials on the left and right elements, respectively. The resulting scheme is a DG method of degree n, or in short notation DG(P_n). By simply increasing the degree n of the polynomials, the DG methods of corresponding higher order are obtained.

B. Efficient Hyperbolic rDG Discretization

Compared with reconstructed FV methods, the DG methods would require more degrees of freedom, additional domain integration, and more Gauss quadrature points for the boundary integration, which leads to more computational costs and storage requirements. Inspired by the reconstructed DG methods from Dumbser et al. in the frame of $\mathrm{P}_n\mathrm{P}_m$ scheme[23–25], termed rDG($\mathrm{P}_n\mathrm{P}_m$) in this paper, least-squares based and variational reconstruction based rDG methods are designed to achieve high order of accuracy while reducing the computational cost. The rDG method is a general framework that contains the FV and DG methods as special cases, thus allowing for a direct efficiency comparison. For rDG($\mathrm{P}_n\mathrm{P}_m$) method with $m > n$, a higher-order reconstructed numerical solution is constructed over an element Ω_i:

$$\mathbf{U}_h^R(x, t, \tau) = \mathbf{C}^R(x)\mathbf{V}^R(t, \tau), \tag{19}$$

where the superscript R indicates reconstructed polynomials, and higher-order derivatives (higher than n-th and up to m-th) are reconstructed from the underlying P_n polynomial. This higher-order numerical solution $\mathbf{U}_h^R$ is used for flux and source term computations in order to raise the order of accuracy. There are three approaches to the reconstruction. One is a least-squares reconstruction method, and another is a variational reconstruction method. The variational reconstruction generates a globally coupled system of equations for gradients by minimizing jumps in the solution and derivatives at element interfaces [19]. The resulting linear system is iteratively solved along with the solution iteration, and therefore the cost is comparable to a least-squares reconstruction. See Ref.[19] for details. The method based on the least-squares/variational reconstruction is expressed by rDG($\mathrm{P}_n\mathrm{P}_m$). The third approach, which is unique in the hyperbolic diffusion formulation considered here, is to directly use the gradient variables and their moments to evaluate the higher-order derivatives in the primary solution polynomial. Or equivalently, this approach can be thought of as defining the primary solution as P_m, and use the higher-order moments to represent the gradient variables in the hyperbolic diffusion formulation. This is the key idea to effectively reducing the number of discrete unknowns despite the increase in the variables in the hyperbolic diffusion formulation. The method based on this approach is expressed by DG($\mathrm{P}_0\mathrm{P}_m$). Naturally, rDG($\mathrm{P}_n\mathrm{P}_m$) and DG($\mathrm{P}_0\mathrm{P}_m$) can be combined to generate efficient schemes as we will discuss later. Below, we describe this efficient construction of hyperbolic rDGV schemes for the combination of DG($\mathrm{P}_0\mathrm{P}_1$) and DG(P_0) in one dimension as an example. Further details can be found in Ref.[11].

By moving the third and fourth terms to the right-hand-side (r.h.s.) in Eq. (17), we arrive at

$$\mathbf{M}_\tau \frac{\partial \mathbf{V}}{\partial \tau} + \mathbf{M}_t \frac{\partial \mathbf{V}}{\partial t} = \mathbf{R}(\mathbf{U}_h^R), \tag{20}$$

where $\mathbf{M}_\tau$ and $\mathbf{M}_t$ are the mass matrices defined as,

$$\mathbf{M}_\tau = \int_{\Omega_i} \mathbf{C}^T \mathbf{C} d\Omega, \quad \mathbf{M}_t = \int_{\Omega_i} \mathbf{C}^T \mathbf{T} \mathbf{C} d\Omega, \tag{21}$$

and $\mathbf{R}$ is the residual vector, defined as

$$\mathbf{R} = \int_{\Omega_i} \frac{\partial \mathbf{C}^T}{\partial x_k} \mathbf{F}_k(\mathbf{U}_h^R) + \int_{\Omega_i} \mathbf{C}^T \mathbf{S}(\mathbf{U}_h^R) d\Omega - \int_{\Gamma_i} \mathbf{C}^T \mathbf{F}_k(\mathbf{U}_h^R) \mathbf{n}_k d\Gamma. \tag{22}$$

The mass matrix $\mathbf{M}_t$ holds the key to the construction of explicit time-accurate schemes as we will discuss later. Boundary conditions are enforced weakly through the numerical flux in a similar manner as in the previous work [26]. Based on different rDG methods, effective hyperbolic rDG schemes can be constructed. The format $\mathbf{A} + \mathbf{B}$ is used to indicate the discretization method for the system, where $\mathbf{A}$ refers to the discretization method for φ and $\mathbf{B}$ refers to the discretization method for its derivatives. Various choices and combinations for $\mathbf{A}$ and $\mathbf{B}$ are compared in Ref.[11]. As an example, consider DG(P_0P_1)+DG(P_0) with

$$\mathbf{V} = \begin{pmatrix} \overline{\varphi} \\ \overline{\varphi_x} \Delta x \end{pmatrix}, \tag{23}$$

$$\mathbf{C} = \begin{pmatrix} B_1 & B_2 \\ 0 & B_1 \Delta x^{-1} \end{pmatrix}. \tag{24}$$

As we can see here, the basis matrix $\mathbf{C}$ has a off-diagonal term to include the connection between φ and its derivatives, leading to a coupled system. Note that the derivatives of φ are determined as solutions to the hyperbolic diffusion system, whereas conventional P_1 DG methods determine them as solutions to discrete equations derived by the weak formulation. See Ref.[11] for various other high-order schemes and the corresponding non-diagonal basis matrices.

C. Explicit Hyperbolic rDG Scheme

We now show that the hyperbolic rDG method leads to a semi-discrete system, for which explicit time-stepping schemes can be applied. Dropping the pseudo time terms in Eq.(20), we obtain the following system of ordinary differential equations in the physical time:

$$\mathbf{M}_t \frac{d\mathbf{V}}{dt} = \mathbf{R}(\mathbf{U}_h^R), \tag{25}$$

where the mass matrix $\mathbf{M}_t$ is given, for DG(P_0P_1)+DG(P_0), as

$$\mathbf{M}_t = \int_{\Omega_i} \mathbf{C}^T \mathbf{T} \mathbf{C} d\Omega = \int_{\Omega_i} \begin{pmatrix} B_1^2 & B_1 B_2 \\ B_1 B_2 & B_2^2 \end{pmatrix} d\Omega = \begin{pmatrix} h & 0 \\ 0 & h^3/12 \end{pmatrix}. \tag{26}$$

Remarkably, a nonzero entry has been created in $\mathbf{M}_t(2,2)$, from $\mathbf{T}$ with $\mathbf{T}(2,2) = 0$, due to the non-diagonal entry in $\mathbf{C}$. That is, a physical time derivative has been created in the second equation for the gradient variable. The system can be integrated in time by explicit time-stepping schemes; the solution and gradient variables are both updated with their own right hand sides. This is a time-accurate semi-discrete system because the pseudo time term has been dropped. Apparently, the time accurate discrete equation for the gradient variable has been derived automatically by the DG differentiation: multiply the first equation (i.e., the diffusion equation) by B_2, and then integrate it. The process is equivalent to differentiation, and it generates a discrete approximation to the spatial derivative of the diffusion equation. But that is not the

whole story. To see what is going on, we fully discretize the semi-discrete system (25) on a uniform grid with a spacing h, using Simpson's rule for the volume integral:

$$\frac{d\overline{\varphi}_i}{dt} = \frac{\nu}{h}\left[(\overline{\mathrm{v}}_x)_{i+1/2} - (\overline{\mathrm{v}}_x)_{i-1/2}\right], \tag{27}$$

$$\frac{d(\overline{\mathrm{v}}_x)_i}{dt} = \nu\frac{(\overline{\mathrm{v}}_x)_{i+1/2} - 2(\overline{\mathrm{v}}_x)_i + (\overline{\mathrm{v}}_x)_{i-1/2}}{(h/2)^2} + \frac{12}{h^2T_r}\left[\frac{\overline{\varphi}_{i+1/2} - \overline{\varphi}_{i-1/2}}{h} - (\overline{\mathrm{v}}_x)_j\right], \tag{28}$$

where $\mathbf{V}_i = (\overline{\varphi}_i, (\overline{\mathrm{v}}_x)_i)$, the subscripts $i-1/2$ and $i+1/2$ indicate the left and right interfaces of a cell i and values from the numerical flux (18). In the second equation, the time derivative and the first term on the right hand side are the approximation to the spatial derivative of the diffusion equation created by the DG differentiation. However, the second term on the right hand side is the one that dominates because of the factor $1/h^2$. Therefore, the semi-discrete system is, in fact, a consistent approximation to the following system:

$$\begin{cases} \dfrac{\partial\varphi}{\partial t} = \nu\dfrac{\partial \mathrm{v}_x}{\partial x}, \\ \dfrac{\partial \mathrm{v}_x}{\partial t} = \dfrac{1}{\epsilon}\left(\dfrac{\partial\varphi}{\partial x} - \mathrm{v}_x\right), \end{cases} \tag{29}$$

where

$$\epsilon = \frac{h^2T_r}{12} = O(h^2), \tag{30}$$

which is small enough to preserve up to second-order accuracy [17,20], and thus the resulting discrete scheme is time accurate.

A few remarks are in order. First, the DG differentiation part is equivalent to a conventional P_1 DG scheme applied to the scalar diffusion equation, and therefore the same order of accuracy cannot be expected for both $\overline{\varphi}_i$ and $(\overline{\mathrm{v}}_x)_i$ on irregular grids; $(\overline{\mathrm{v}}_x)_i$ would be one order lower. The same order of accuracy observed in numerical experiments supports the above argument: the leading term is not $\nu\partial_{xx}\mathrm{v}_x$ but $\frac{1}{\epsilon}\left(\frac{\partial\varphi}{\partial x} - \mathrm{v}_x\right)$. In other words, the strong coupling in the differential-equation-level is retained in the discrete level, and the scheme is solving the hyperbolic diffusion system with a sufficiently small relaxation parameter ϵ. Although similar, the above scheme is different from those of Montecinos and Toro [17,20] in the numerical flux (18), where our T_r is $O(1)$ while their T_r is $O(h^{1+r/2})$. The latter makes the dissipation coefficient $O(1/h)$ for second-order schemes, and it looks very much like conventional diffusion schemes. Secondly, the time accurate semi-discrete system is a feature unique to the rDG/DG methods; it cannot happen in FV methods nor in the schemes of Montecinos and Toro [17,20], which is not based on the weak formulation (thus the relaxation time needed to be adjusted directly). Furthermore, it is a unique feature coming from using the non-diagonal basis matrix $\mathbf{C}$ that arises from the efficient construction as a test function, where the gradient variable is used to form a higher-order polynomial for φ [11]. The feature may be lost if a different test function is chosen, e.g., the diagonal part of $\mathbf{C}$. Finally, the above argument is specific to DG($\mathrm{P}_0\mathrm{P}_1$)+DG(P_0), and it remains to be extended to higher-order schemes. In this paper, the time consistency of the semi-discrete system (25) in higher-order schemes is verified by numerical experiments.

To demonstrate the semi-discrete system (25) for unsteady problems, in this paper, we employ the following explicit three-stage third-order TVD Runge-Kutta scheme [27,28](TVDRK3) for time integration:

$$\begin{cases} \mathbf{V}^{(1)} = \mathbf{V}^n + \Delta t{\mathbf{M}_t}^{-1}\mathbf{R}(\mathbf{V}^n), \\ \mathbf{V}^{(2)} = \dfrac{3}{4}\mathbf{V}^n + \dfrac{1}{4}[\mathbf{U}^{(1)} + \Delta t{\mathbf{M}_t}^{-1}\mathbf{R}(\mathbf{V}^{(1)})], \\ \mathbf{V}^{n+1} = \dfrac{1}{3}\mathbf{V}^n + \dfrac{2}{3}[\mathbf{V}^{(2)} + \Delta t{\mathbf{M}_t}^{-1}\mathbf{R}(\mathbf{V}^{(2)})]. \end{cases} \tag{31}$$

In summary, explicit hyperbolic DG schemes can be constructed in a straightforward manner because the Galerkin formulation creates a physical time derivative for the gradient variable v_x, and results in a semi-discrete system ready for explicit time stepping. Further details such as stability properties remain to be investigated; here, we demonstrate numerically that high-order explicit schemes actually work and deliver design orders of accuracy for unsteady problems.

IV. Numerical Examples

A. 1D unsteady heat equation

Consider the following 1D heat conduction problem:

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} = \nu \dfrac{\partial^2 \varphi}{\partial x^2} \quad x \in [0,1] \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = C\sin(\pi x) \end{cases}. \tag{32}$$

with

$$\nu = 0.06, \quad C = 50. \tag{33}$$

The exact solution is given by

$$\varphi(x,t) = C\exp(-\nu\pi^2 t)\sin(\pi x). \tag{34}$$

In this case, three different hyperbolic rDG schemes, DG(P_0P_1)+DG(P_0), DG(P_0P_2)+rDG-LS(P_0P_1), DG(P_0P_2)+rDG-VR(P_0P_1), are used to perform computations up to $t = 1$, with a fixed time step $\Delta = 10^{-4}$. DG(P_0P_1)+DG(P_0) is a first-order scheme while DG(P_0P_2)+rDG-LS(P_0P_1) and DG(P_0P_2)+rDG-VR(P_0P_1) are second-order schemes [11, 26]. Error convergence results are shown in Figure 1, with uniform grids with 8, 16, 32, and 64 elements. Clearly, the design order of accuracy has been confirmed for all schemes. Furthermore, the same order of accuracy has been achieved in the gradient variable for all schemes.

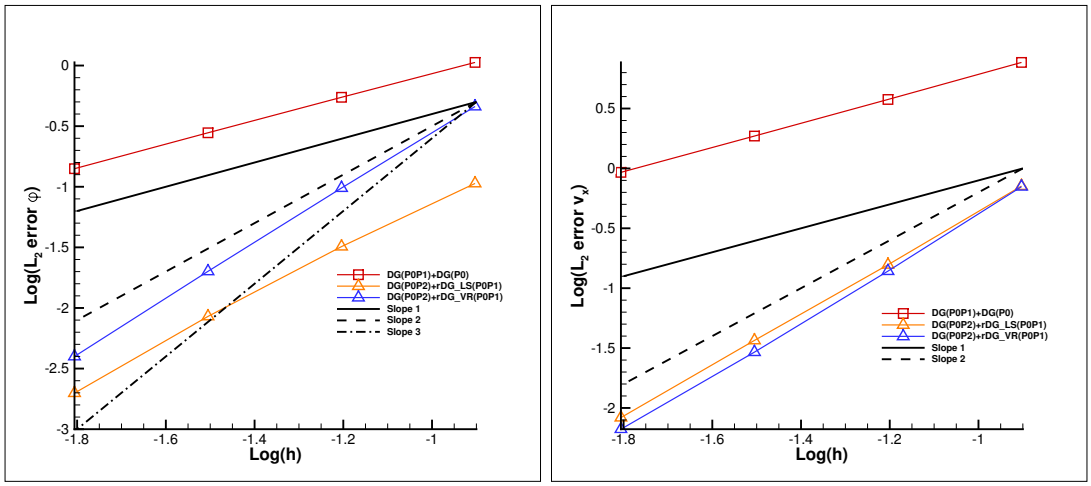

Figure 1: Grid refinement study on regular grids for 1D unsteady heat equation.

B. 1D unsteady advection diffusion problem

Consider the following exact solution to the 1D unsteady linear advection diffusion problem

$$\frac{\partial \varphi}{\partial t} + a\frac{\partial \varphi}{\partial x} = \nu \frac{\partial^2 \varphi}{\partial x^2}, \qquad x \in [0,2] \tag{35}$$

with the exact solution given as

$$\varphi(x,t) = \frac{1}{\sqrt{4t+1}} \exp\left(-\frac{(x-at-x_0)^2}{\nu(4t+1)}\right), \tag{36}$$

where

$$a = 10^4, \quad \nu = 0.01, \quad x_0 = 0.5. \tag{37}$$

The parameters are chosen to yield an advection dominant problem. The initial Gaussian bump will travel with a constant velocity with a small diffusion effect. The numerical flux is constructed as a sum of the

upwind diffusion flux (18) and an upwind advective flux as described in Ref.[29]. For this test case, we test the fourth-order scheme, DG(P_0P_3)+rDG-VR(P_0P_2), in addition to the three schemes considered before. These schemes are known to achieved one-order higher-order of accuracy in the primal solution for advection-dominated cases [11]. A grid refinement test has been carried out to verify the spatial order of accuracy in this unsteady case. A small physical time step $\Delta t = 10^{-9}$ has been set for all the grids with a fixed final time $t_{\text{end}} = 10^{-4}$. The grids are uniform with $nelem = 32, 64, 128$ and 256. Dirichlet boundary conditions have been applied on both ends. All the presented methods use the same degrees of freedom, 2, which is equivalent to a conventional P_1 DG method. Note also that a FV scheme would have 2 unknowns per element for the hyperbolic diffusion formulation; however, the developed hyperbolic rDG schemes are not FV schemes because of the coupling generated by the non-diagonal basis matrix $\mathbf{C}^T$. The numerical results are shown in Figure 2. As can be observed, DG(P_0P_1)+DG(P_0) achieves first-order accuracy in the gradient and one-order higher-order of accuracy in the primal solution as expected. Unexpectedly, DG(P_0P_2)+rDG-LS(P_0P_1) and DG(P_0P_2)+rDG-VR(P_0P_1) achieve asymptotically second-order accuracy in the gradient and fourth-order accuracy in the primal solution. Furthermore, DG(P_0P_3)+rDG-VR(P_0P_2) gives fourth-order accuracy in both the solution and the gradient.

Secondly, a comparison between the developed schemes and a conventional DG (Direct DG [30] in this study), is shown in Figure 3. Here, periodic boundary conditions are enforced with $\Delta t = 10^{-9}$, $t_{\text{end}} = 10^{-3}$ on a uniform mesh ($nelem = 32$). Clearly, the presented hyperbolic rDG schemes can outperform the conventional counterpart, better resolving the peak of the Gaussian profile. Efficiency of the hyperbolic rDG schemes are evident from the fact that all schemes use 2 degrees of freedom per element, including the P_1 DDG.

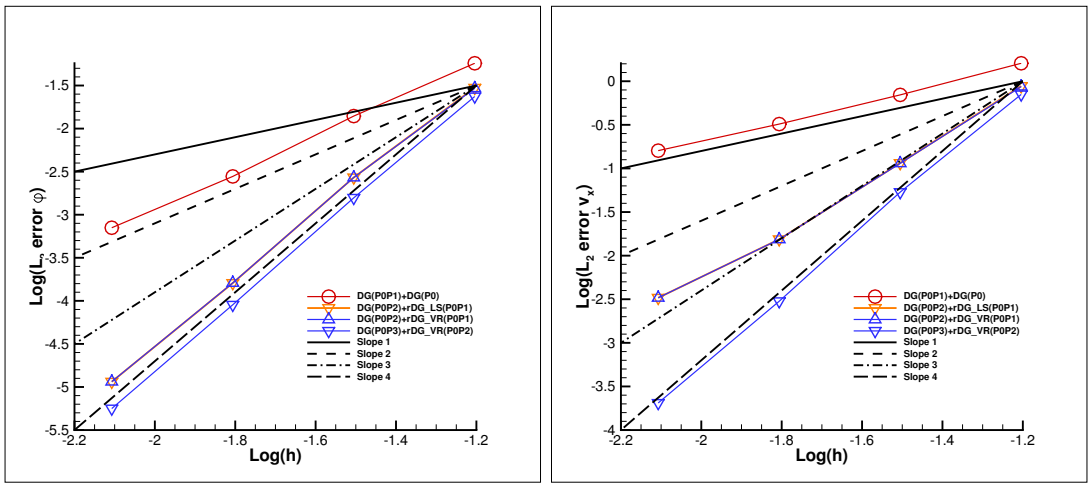

Figure 2: Grid refinement study on regular grids for 1D unsteady case.

C. 2D unsteady advection diffusion problem

In this case, we extend the previous case to two dimensions. The analytical solution is given by

$$\varphi(x,y,t) = \frac{1}{4t+1}\exp\left(-\frac{(x-at-x_0)^2+(y-bt-y_0)^2}{\nu(4t+1)}\right), \quad (x,y)\in[0,2]\times[0,2], \tag{38}$$

where

$$x_0 = y_0 = 1.0, \quad a = b = 10^{-5}, \quad \nu = 0.01. \tag{39}$$

The following schemes are tested and compared:

$$\begin{aligned}
&\text{DG}(P_0P_1)\text{+DG}(P_0), \quad \text{and} \quad \text{DG}(P_0P_2)\text{+DG}(P_1),\\
&\text{DG}(P_0P_2)\text{+rDG-LS}(P_0P_1), \quad \text{and} \quad \text{DG}(P_0P_3)\text{+rDG-LS}(P_1P_2),\\
&\text{DG}(P_0P_2)\text{+rDG-VR}(P_0P_1), \quad \text{and} \quad \text{DG}(P_0P_3)\text{+rDG-VR}(P_0P_2),\\
&\text{DG}(P_0P_3)\text{+rDG-VR}(P_1P_2), \quad \text{and} \quad \text{DG}(P_0P_4)\text{+rDG-VR}(P_1P_3).
\end{aligned}$$

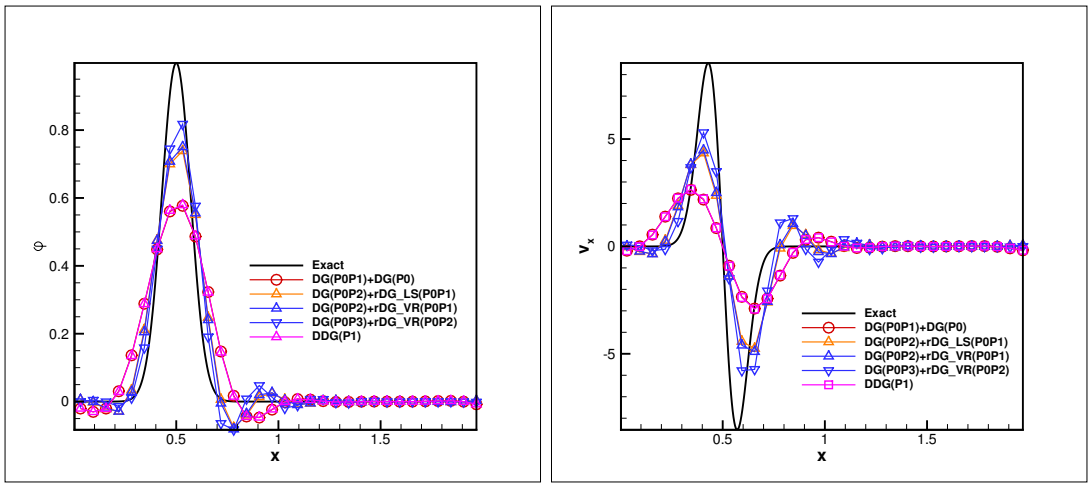

Figure 3: Grid refinement study on regular grids for 1D unsteady case.

Note that the hyperbolic rDG schemes with rDG(P_1P_2) and rDG(P_1P_3) use six degrees of freedom per element, which is equivalent to the P_2 DG scheme, DG(P_0P_2)+DG(P_1). Two sets of triangular meshes are used in the test, namely regular and irregular grids. The sample of each type of grids are shown in Figure 4.

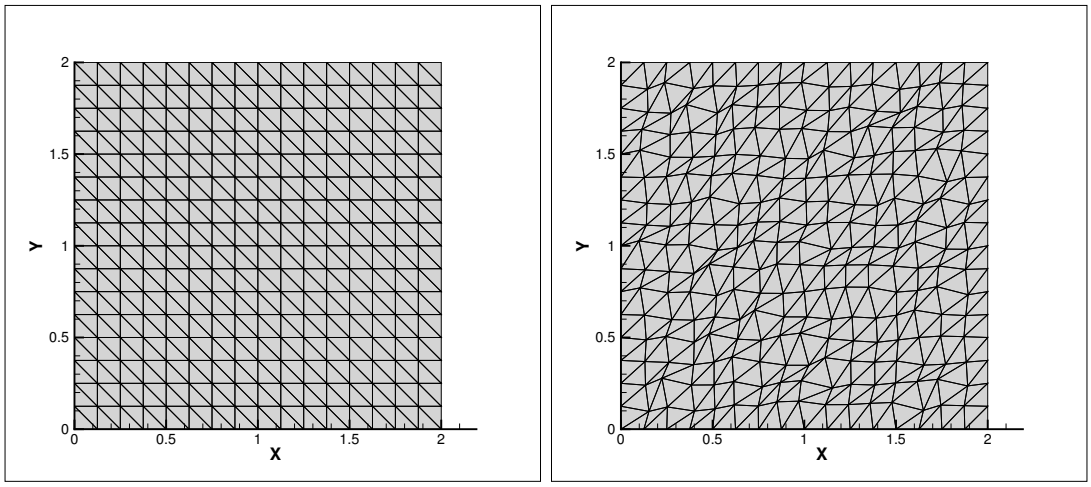

Figure 4: The sample regular and irregular grids, generated from a 16×16 Cartesian grid by subdivision and nodal perturbation.

The physical time step is set as $\Delta t = 10^{-3}$ with $t_{end} = 1$. Dirichlet boundary conditions are applied on all the boundary faces. The numerical results are shown in Table 1 and Figures 5 to 6. All the presented schemes are shown to provide the design or higher order of accuracy in this unsteady case. In particular, DG(P_0P_3)+rDG-VR(P_0P_2) achieves slightly higher than fourth-order accuracy for both the solution and the gradient on irregular grids with only three degrees of freedom per element, which is equivalent to a conventional P_1 DG scheme that is second-order in the solution and first-order in the gradient.

V. Conclusions and Outlook

Contrary to expectations, we have demonstrated that rDG methods applied to a hyperbolic diffusion system are time-accurate with explicit time-stepping schemes. The hyperbolic diffusion formulation used here is formally equivalent to the diffusion equation only in the steady state. However, it has been discovered that the reconstructed discontinuous Galerkin discretization automatically creates a valid time-accurate

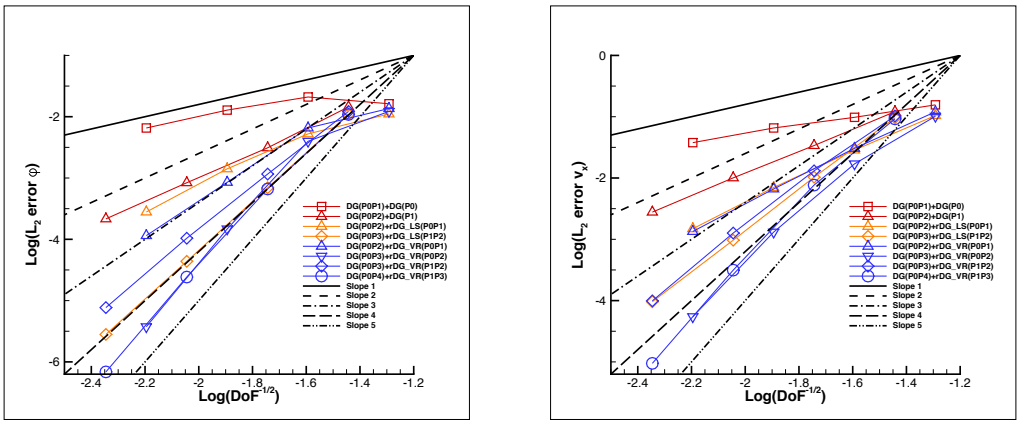

Figure 5: Grid refinement study on regular grids for 2D unsteady case.

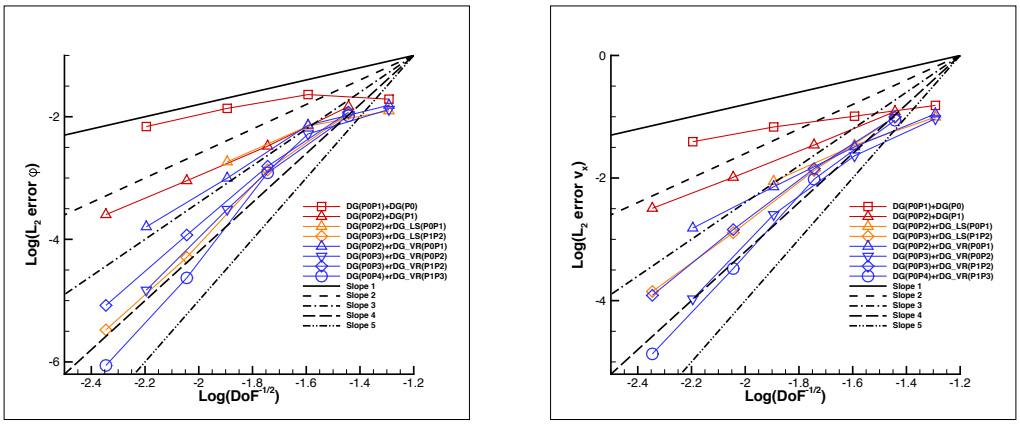

Figure 6: Grid refinement study on irregular grids for 2D unsteady case.

semi-discrete system that can be integrated in time by explicit time-stepping schemes. This special property has been found due to the non-diagonal nature of the basis matrix used in the efficient construction of the hyperbolic rDG method. The off-diagonal entries create the physical time derivatives through the weak formulation, and also scale the relaxation properly to recover time accuracy. In effect, no special technique nor modification is necessary; explicit time-stepping schemes can be constructed in a rather straightforward manner. Several high-order hyperbolic rDG schemes have been constructed and demonstrated for an unsteady advection-diffusion equation. These schemes are very efficient, achieving up to fourth-order accuracy on irregular grids with only three or six degrees of freedom per element in two dimensions, while conventional DG schemes can achieve only up to third-order accuracy with six degrees of freedom per element. Also, the developed hyperbolic rDG schemes achieve the same order of accuracy in the solution and the gradient on irregular grids in the diffusion limit while conventional DG schemes yield one-order lower order of accuracy in the gradient. These hyperbolic rDG schemes provide attractive alternatives to solve unsteady problems with explicit time-stepping schemes. An undergoing effort is being put on extending the method to nonlinear equations to show its potential for applications to the Navier-Stokes equation on fully 3D unstructured grids.

Acknowledgments

This work has been funded by the U.S. Army Research Office under the contract/grant number W911NF-16-1-0108 with Dr. Matthew Munson as the program manager.

Table 1: Order of accuracy on 2D unsteady advection diffusion problem.

Scheme [DoFs]	Regular		Irregular	
	φ	v_x	φ	v_x
DG($\mathrm{P}_0\mathrm{P}_1$)+DG(P_0) [3]	0.97	0.79	0.99	0.80
DG($\mathrm{P}_0\mathrm{P}_2$)+DG(P_1) [6]	1.96	1.86	1.84	1.68
DG($\mathrm{P}_0\mathrm{P}_2$)+rDG-LS($\mathrm{P}_0\mathrm{P}_1$) [3]	2.34	2.24	1.94	1.95
DG($\mathrm{P}_0\mathrm{P}_3$)+rDG-LS($\mathrm{P}_1\mathrm{P}_2$) [6]	3.96	3.34	3.93	3.18
DG($\mathrm{P}_0\mathrm{P}_2$)+rDG-VR($\mathrm{P}_0\mathrm{P}_1$) [3]	2.89	2.30	2.64	2.24
DG($\mathrm{P}_0\mathrm{P}_3$)+rDG-VR($\mathrm{P}_0\mathrm{P}_2$) [3]	5.30	4.57	4.38	4.57
DG($\mathrm{P}_0\mathrm{P}_3$)+rDG-VR($\mathrm{P}_1\mathrm{P}_2$) [6]	3.76	3.66	3.81	3.56
DG($\mathrm{P}_0\mathrm{P}_4$)+rDG-VR($\mathrm{P}_1\mathrm{P}_3$) [6]	5.14	5.06	4.74	4.62

References

[1]H. Nishikawa. A first-order system approach for diffusion equation. i: Second-order residual-distribution schemes. *Journal of Computational Physics*, 227(1):315–352, 2007.

[2]H. Luo, L. Luo, R. Nourgaliev, and V. Mousseau. A Reconstructed Discontinuous Galerkin Method for the Compressible Euler Equations on Arbitrary Grids. *AIAA Paper*, AIAA-2009-3788, 2009.

[3]H. Luo, L. Luo, R. Nourgaliev, V. Mousseau, and N. Dinh. A Reconstructed Discontinuous Galerkin Method for the Compressible Navier-STokes Equations on Arbitrary Grids. *Journal of Computational Physics*, 229(19):6961–6978, 2010.

[4]H. Luo, Y. Xia, and R. Nourgaliev. A Class of Reconstructed Discontinuous Galerkin Methods in Computational Fluid Dynamics. *International Conference on Mathematics and Computational Methods Applied to Nuclear Science and Engineering (M&C2011)*, Brazil, 2011.

[5]H. Luo, L. Luo, A Ali, R. Nourgaliev, and C. Cai. A Parallel, Reconstructed Discontinuous Galerkin Method for the Compressible Flows on Arbitrary Grids. *Communication in Computational Physics*, 9(2):363–389, 2011.

[6]H. Luo, Y. Xia, S. Li, and R. Nourgaliev. A Hermite WENO Reconstruction-Based Discontinuous Galerkin Method for the Euler Equations on Tetrahedral grids. *Journal of Computational Physics*, 231(16):5489–5503, 2012.

[7]H. Luo, Y. Xia, S. Spiegel, R. Nourgaliev, and Z. Jiang. A reconstructed discontinuous Galerkin method based on a hierarchical WENO reconstruction for compressible flows on tetrahedral grids. *Journal of Computational Physics*, 236:477–492, 2013.

[8]Jialin Lou, Hong Luo, , and Hiroaki Nishikawa. Discontinuous Galerkin methods for hyperbolic advection-diffusion equation on unstructured grids. In *Proc. of The 9th International Conference on Computational Fluid Dynamics*, Istanbul, Turkey, 2016.

[9]Jialin Lou, Xiaodong Liu, Hong Luo, and Hiroaki Nishikawa. Reconstructed discontinuous Galerkin methods for hyperbolic diffusion equations on unstructured grids. In *55th AIAA Aerospace Sciences Meeting*, AIAA Paper 2017-0310, Grapevine, Texas, 2017.

[10]Jialin Lou, Lingquan Li, Hong Luo, and Hiroaki Nishikawa. First-order hyperbolic system based reconstructed discontinuous Galerkin methods for nonlinear diffusion equations on unstructured grids. In *2018 AIAA Aerospace Sciences Meeting*, AIAA Paper 2018-2094, 2018.

[11]Jialin Lou, Lingquan Li, Hong Luo, and Hiroaki Nishikawa. Reconstructed discontinuous Galerkin methods for linear advection-diffusion equations based on first-order hyperbolic system. *J. Comput. Phys.*, 369:103–124, 2018.

[12]Lingquan Li, Jialin Lou, Hong Luo, and Hiroaki Nishikawa. A new hyperbolic Navier-Stokes formulation and efficient reconstructed discontinuous Galerkin discretizations. In *48th AIAA AIAA Fluid Dynamics Conference*, AIAA Paper, Atlanta, GA, 2018. to be published as AIAA paper.

[13]Hiroaki Nishikawa and Philip L Roe. Third-order active-flux scheme for advection diffusion: Hyperbolic diffusion, boundary condition, and Newton solver. *Computers & Fluids*, 125:71–81, 2016.

[14]Yi Liu and Hiroaki Nishikawa. Third-order inviscid and second-order hyperbolic Navier-Stokes solvers for three-dimensional unsteady inviscid and viscous flows. In *55th AIAA Aerospace Sciences Meeting*, AIAA Paper 2017-0738, Grapevine, Texas, 2017.

[15]C Cattaneo. A form of heat conduction equation which eliminates the paradox of instantaneous propagation. *Compte Rendus*, 247(4):431–433, 1958.

[16]Pierre Vernotte. Les paradoxes de la theorie continue de l'equation de la chaleur. *Compt. Rendu*, 246:3154–3155, 1958.

[17]Eleuterio F Toro and Gino I Montecinos. Advection-diffusion-reaction equations: hyperbolization and high-order ader discretizations. *SIAM Journal on Scientific Computing*, 36(5):A2423–A2457, 2014.

[18]J. Cheng, T. Liu, and H. Luo. A hybrid reconstructed discontinuous Galerkin method for compressible flows on arbitrary grids. *Computers & Fluids*, 139:68–79, 2016.

[19]Q. Wang, Y. Ren, J. Pan, and W. Li. Compact high order finite volume method on unstructured grids III: Variational reconstruction. *Journal of Computational Physics*, 337:1–26, 2017.

[20]Gino I Montecinos, Lucas O Müller, and Eleuterio F Toro. Hyperbolic reformulation of a 1d viscoelastic blood flow model and ader finite volume schemes. *Journal of Computational Physics*, 266:101–123, 2014.

[21]H. Nishikawa. First, second, and third order finite-volume schemes for Navier-Stokes equations. In *Proc. of 7th AIAA Theoretical Fluid Mechanics Conference, AIAA Aviation and Aeronautics Forum and Exposition 2014*, AIAA Paper 2014-2091, Atlanta, GA, 2014.

[22]H. Nishikawa. First-, second-, and third-order finite-volume schemes for diffusion. *J. Comput. Phys.*, 256:791–805, 2014.

[23]M. Dumbser, M. Kaser, V. A. Titarev, and E. F. Toro. Quadrature-free non-oscillatory finite volume schemes on unstructured meshes for nonlinear hyperbolic systems. *Journal of Computational Physics*, 226(1):204–243, 2007.

[24]M. Dumbser and O. Zanotti. A unified framework for the construction of one-step finite volume and discontinuous Galerkin schemes on unstructured meshes. *Journal of Computational Physics*, 227(18):8209–8253, 2008.

[25]M. Dumbser. Arbitrary High Order PNPM Schemes on Unstructured Meshes for the Compressible Navier-Stokes Equations. *Computers & Fluids*, 39(1):60–76, 2010.

[26]J. Lou, X. Liu, H. Luo, and H. Nishikawa. Reconstructed discontinuous Galerkin methods for hyperbolic diffusion equations on unstructured grids. In *55th AIAA Aerospace Sciences Meeting*, AIAA Paper 2017-0310, Grapevine, Texas, 2017.

[27]B. Cockburn, G. Karniadakis, and Shu C. W. The Development of Discontinuous Galerkin Method. *Discontinuous Galerkin Methods, Theory, Computation, and Applications. Edited by B. Cockburn, G. E. Karniadakis, and C. W. Shu. Lecture Notes in Computational Science and Engineering*, 11:5–50, 2000.

[28]B. Cockburn and C. W. Shu. The Runge-Kutta Discontinuous Galerkin Method for conservation laws V: Multidimensional System. *Journal of Computational Physics*, 141:199–224, 1998.

[29]H. Nishikawa. First, second, and third order finite-volume schemes for advection–diffusion. *Journal of Computational Physics*, 273:287–309, 2014.

[30]Hailiang Liu and Jue Yan. The direct discontinuous Galerkin (DDG) methods for diffusion problems. *SIAM Journal on Numerical Analysis*, 47(1):675–698, 2009.