

Reconstructed Discontinuous Galerkin Methods for Hyperbolic Diffusion Equations on Unstructured Grids

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Reconstructed Discontinuous Galerkin (rDG) methods are presented for solving diffusion equations based on a first-order hyperbolic system (FOHS) formulation. The idea is to combine the advantages of the FOHS formulation and the rDG methods in an effort to develop a more reliable, accurate, efficient, and robust method for solving the diffusion equations. The developed hyperbolic rDG methods can be made to have higher-order accuracy than conventional DG methods with fewer degrees of freedom. A number of test cases for different diffusion equations are presented to assess accuracy and performance of the newly developed hyperbolic rDG methods in comparison with the standard BR2 DG method. Numerical experiments demonstrate that the hyperbolic rDG methods are able to achieve the designed optimal order of accuracy for both solutions and their derivatives on regular, irregular, and heterogeneous girds, and outperform the BR2 method in terms of the magnitude of the error, the order of accuracy, the size of time steps, and the CPU times required to achieve steady state solutions, indicating that the developed hyperbolic rDG methods provide an attractive and probably an even superior alternative for solving the diffusion equations.

I. Introduction

The discontinuous Galerkin (DG) methods^{2,4,5,8-10,16,18,19,22-26,41,42,45} have recently become popular for the solution of systems of conservation laws. Nowadays, they are widely used in computational fluid dynamics, computational acoustics, and computational magneto-hydrodynamics. The DG methods combine two advantageous features commonly associated to the finite element (FE) and finite volume (FV) methods. As in classical finite element methods, the order of accuracy is obtained by means of high-order polynomial approximation within an element rather than by wide stencils as in the finite volume methods. The physics of wave propagation is, however, accounted for by solving the Riemann problems that arise from the discontinuous representation of the solution at element interfaces. In this respect, the DG methods (DGMs) are therefore similar to the finite volume methods. The DG methods have many attractive features: 1) They have several useful mathematical properties with respect to conservation, stability, and convergence; 2) They can be easily extended to higher-order (>2nd) approximation; 3) They are well suited for complex geometries since they can be applied on unstructured grids. In addition, the methods can also handle non-conforming elements, where the grids are allowed to have hanging nodes; 4) The methods are highly parallelizable, as they are compact and each element is independent. Since the elements are discontinuous, and the inter-element communications are minimal, domain decomposition can be efficiently employed; 5) They can easily handle adaptive strategies, since refining or coarsening a grid can be achieved without considering the continuity restriction commonly associated with the conforming elements. The methods allow easy implementation of

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hp-refinement, for example, the order of accuracy, or shape, can vary from element to element. However, the DGMs have a number of their own weaknesses. In particular, how to effectively control spurious oscillations in the presence of strong discontinuities, how to reduce the computing costs for the DGMs, and how to efficiently solve elliptic problems or discretize diffusion terms in the parabolic equations remain the three unresolved and challenging issues in the DGMs.

The DGMs have been recognized as expensive in terms of both computational costs and storage requirements. Indeed, compared to the FE and FV methods, the DGMs require solutions of systems of equations with more unknowns for the same grids. In order to reduce high costs associated with the DGMs, Dumbser et al.¹²⁻¹⁴ have introduced a new family of reconstructed DGM, termed PnPm schemes and referred to as rDG(PnPm) in this paper, where Pn indicates that a piecewise polynomial of degree of n is used to represent a DG solution, and Pm represents a reconstructed polynomial solution of degree of m $(m \ge n)$ that is used to compute the fluxes. The rDG(PnPm) schemes^{27,28,31,46,47} are designed to enhance the accuracy of the DGM by increasing the order of the underlying polynomial solution. The beauty of rDG(PnPm) schemes is that they provide a unified formulation for both FVM and DGM, and contain both classical FVM and standard DGM as two special cases of rDG(PnPm) schemes. When n = 0, i.e. a piecewise constant polynomial is used to represent a numerical solution, rDG(P0Pm) is nothing but classical high order FV schemes, where a polynomial solution of degree $m \ (m \ge 1)$ is reconstructed from a piecewise constant solution. When m = n, the reconstruction reduces to the identity operator, and rDG(PnPn) scheme yields a standard DG(Pn) method. For n > 0, and m > n, a new family of numerical methods from third-order of accuracy upwards is obtained. A Hierarchical WENO-based rDG method^{30,32} is designed not only to reduce the high computing costs of the DGM, but also to avoid spurious oscillations in the vicinity of strong discontinuities. thus effectively overcoming the first two shortcomings of the DG methods.

The DGMs are indeed a natural choice for the solution of the hyperbolic equations, such as the compressible Euler equations. However, the DG formulation is far less certain and advantageous for elliptic problems or parabolic equations such as the compressible Navier-Stokes equations, where diffusive fluxes exist and which require the evaluation of the solution derivatives at the interfaces. Taking a simple arithmetic mean of the solution derivatives from the left and right is inconsistent, because it does not take into account a possible jump of the solutions. A number of numerical methods have been proposed in the literature to address this issue, such as those by Bassi and Rebay,^{3, 5, 6} Cockburn and Shu,¹¹ Baumann and Oden,⁷ Peraire and Persson,⁴⁰ and many others. Arnold et al.¹ have analyzed a large class of DGM for second-order elliptic problems in a unified formulation. All these methods have introduced in some way the influence of the discontinuities in order to define correct and consistent diffusive fluxes. Gassner et al.¹⁷ introduced a numerical scheme based on the exact solution of the diffusive generalized Riemann problem for the DGM. Liu et al.^{?,?} proposed a direct discontinuous Galerkin (DDG) method to solve diffusion problems based on the direct weak formulation for solutions of parabolic equations. Cheng et al.? successfully extended the DDG method to solve the compressible Navier-Stokes equations on arbitrary grids. Luo et al. have developed a reconstructed discontinuous Galerkin method using an inter-cell reconstruction²⁹ for the solution of the compressible Navier-Stokes equations. Unfortunately, all these methods seem to require substantially more computational effort than the classical continuous finite element methods, which are naturally more suited for the discretization of elliptic problems.

An alternative approach for viscous discretization is to reformulate the viscous terms as a hyperbolic system using the first-order hyperbolic system (FOHS) formulation developed by Nishikawa³⁵⁻³⁹ over the last several years, and recently extended to the three-dimensional Navier-Stokes equations.^{20,34} In the FOHS formulation, the diffusion equations are first formulated as a first-order system (FOS) by including derivative quantities as additional variables. The system is then rendered to be hyperbolic, which is the distinguished feature of the FOHS method from other FOS methods, by adding pseudo time derivatives to the first-order system. It thus generates a system of pseudo-time evolution equations for the solution and the derivatives in the partial differential equation (PDE) level, not in the discretization level as in DGM. The hyperbolic formulation in the PDE level allows a dramatic simplification in the discretization because well-established methods for hyperbolic systems can be directly applied to the viscous terms. In the pseudo steady state or simply by dropping the pseudo-time derivative after the discretization, a consistent and superior discretization of the diffusion equations is obtained. The FOHS method is especially attractive in the context of the DGM, since it allows the use of inviscid algorithms for the viscous terms and thus greatly simplifies the discretization of the compressible Navier Stokes equations. Moreover, schemes constructed based on the hyperbolic method are known to achieve the same order of accuracy in the solution and the

derivatives on irregular grids; conventional DGMs typically yield one-order-lower accuracy in the derivatives on such grids. This property is attractive since irregular grids are hard to avoid in grid adaptation, which is considered as one of the essential elements for high-order methods to be practical,⁴⁴ and the main target quantities (e.g., the viscous stresses) are derivatives in viscous problems.

The hyperbolic approach applied to a DG discretization method for advection-diffusion equation is discussed in literature,³³ where the idea of Scheme-II³⁷ is extended to construct a hyperbolic DG discretization that minimizes the additional cost associated with the increased number of equations. This specific method achieves the same order of accuracy in the advective term and the gradients as a conventional DG method for the same total number of degrees of freedom. However, the order of accuracy for the diffusive term is one order lower than the conventional DG method (see Table 3 in Ref.³³); it leads to lower order accuracy, for example, in boundary layers. Moreover, since a direct solver is employed for solving the linear system in the Newton method, convergence acceleration by the elimination of second derivatives, which is one of the advantages of the hyperbolic method, is not achieved. Therefore, this approach is, although more efficient than a straightforward DG discretization of the FOHS, actually less efficient than conventional DG methods, not fully taking advantage of the hyperbolic method. Note also that the method in Ref.³³ employs a dissipative Rusanov-type flux, not the upwind hyperbolic-diffusion flux demonstrated in the previous studies for finite-volume methods. As we will show, the latter yields more accurate solutions and gradients. Furthermore, this paper focuses on pure diffusion problems, which are not addressed in Ref.,³³ including a tensor diffusion coefficient, and explores various hyperbolic DG constructions. Developed schemes will be compared with the BR2 diffusion scheme¹⁵ that is widely employed in the DGMs for diffusion and the Navier-Stokes equations;^{16,29} Ref.³³ provides comparisons only with a less popular non-optimal interior penalty scheme. As will be shown, some of the developed DG schemes here achieve one order higher accuracy in the primary variable than that in the auxiliary variables; these schemes indicate a potential for overcoming the lower order accuracy issue for the diffusion term in the method of Ref.³³

The objective of the effort discussed in this paper is to develop a higher-order rDG method for solving diffusion equations based on the first-order hyperbolic system formulation, termed hyperbolic rDG methods in this paper. The idea behind the hyperbolic rDG methods is to combine the advantages of the FOHS formulation and the rDG methods in an effort to develop a more reliable, accurate, efficient, and robust method for solving the diffusion equations and ultimately the incompressible and compressible Navier-Stokes equations on fully irregular, adaptive, anisotropic, unstructured grids. By using the solution derivatives handily available in the FOHS formulation and Taylor basis in the rDG formulation, the hyperbolic rDG methods can be designed to have the same number of degrees-of-freedom as the conventional rDG methods. A number of test cases for different diffusion equations are presented to assess accuracy and performance of the newly developed hyperbolic rDG methods in comparison with the standard BR2 DG method. Numerical experiments demonstrate that the hyperbolic rDG methods are able to achieve the designed optimal order of accuracy for both solutions and their derivatives on regular, irregular, and heterogeneous girds, and outperform the BR2 method in terms of the magnitude of the error, the order of accuracy, the size of time steps, and the CPU times required to achieve steady state solutions, indicating that the developed hyperbolic rDG methods provide an attractive and probably an even superior alternative for solving the diffusion equations.

The remainder of this paper is organized as follows. A FOHS formulation for diffusion equations is described in Section II. The rDG methods for solving the hyperbolic diffusion equations are presented in Section III. Extensive numerical experiments are reported in Section IV. Concluding remarks are given in Section V.

II. First-Order Hyperbolic System formulation

The FOHS formulation³⁵ was introduced by Nishikawa as a radical approach for solving diffusion equation. Consider the following diffusion equation in two dimensions

$$-\nabla \cdot (\mathbf{D}\nabla \varphi) = f, \tag{1}$$

where φ denotes a scalar function, which can be referred to as velocity potential, and **D** denotes a symmetric positive definite diffusion tensor, which can be expressed as

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}_{xx}(x,y) & \mathbf{D}_{xy}(x,y) \\ \mathbf{D}_{xy}(x,y) & \mathbf{D}_{yy}(x,y) \end{bmatrix}.$$
(2)

By introducing an auxiliary vector variable, $\mathbf{V} = (\mathbf{v}_x \ \mathbf{v}_y)$, which is the gradient of the velocity potential, i.e., velocity vector, Eq. (1) can be rewritten as a first-order system of equations as

$$\begin{pmatrix}
-\frac{\partial}{\partial x}(D_{xx}v_x + D_{xy}v_y) - \frac{\partial}{\partial y}(D_{xy}v_x + D_{yy}v_y) = f \\
v_x = \frac{\partial \varphi}{\partial x} \\
v_y = \frac{\partial \varphi}{\partial y}
\end{cases}$$
(3)

This formulation is known for decades as the mixed formulation in the FE community and widely used in many discretization methods including the DG methods. For example, Bassi and Rebay derived the well-known and widely used BR1 and BR2 methods from this formulation.^{3,6} Note that the above system, i.e., Eq. (3) is elliptic in space, which is equivalent to the original equation. In the FOHS formulation, the system is made hyperbolic by adding pseudo-time derivatives with respect to all variables as follows:

$$\begin{cases} \frac{\partial\varphi}{\partial\tau} = \frac{\partial}{\partial x} (\mathbf{D}_{xx}\mathbf{v}_x + \mathbf{D}_{xy}\mathbf{v}_y) + \frac{\partial}{\partial y} (\mathbf{D}_{xy}\mathbf{v}_x + \mathbf{D}_{yy}\mathbf{v}_y) + f \\ \frac{\partial\mathbf{v}_x}{\partial\tau} = \frac{1}{T_r} \left(\frac{\partial\varphi}{\partial x} - \mathbf{v}_x\right) \\ \frac{\partial\mathbf{v}_y}{\partial\tau} = \frac{1}{T_r} \left(\frac{\partial\varphi}{\partial y} - \mathbf{v}_y\right) \end{cases}$$
(4)

where τ is the pseudo time and T_r is a free parameter called the relaxation time. In the FOHS formulation, our interest is to obtain the steady-state solution of the pseudo-time system, i.e., Eq. (4), which is the solution to the original diffusion equation (1). The FOHS can be written in vector form as

$$\frac{\partial \mathbf{U}}{\partial \tau} = \frac{\partial \mathbf{F}_x}{\partial x} + \frac{\partial \mathbf{F}_y}{\partial y} + \mathbf{S},\tag{5}$$

where

$$\mathbf{U} = \begin{bmatrix} \varphi \\ \mathbf{v}_x \\ \mathbf{v}_y \end{bmatrix}, \mathbf{F}_x = \begin{bmatrix} -\mathbf{D}_{xx}\mathbf{v}_x - \mathbf{D}_{xy}\mathbf{v}_y \\ -\varphi/T_r \\ 0 \end{bmatrix}, \mathbf{F}_y = \begin{bmatrix} -\mathbf{D}_{xy}\mathbf{v}_x - \mathbf{D}_{yy}\mathbf{v}_y \\ 0 \\ -\varphi/T_r \end{bmatrix}, \mathbf{S} = \begin{bmatrix} f \\ -\mathbf{v}_x/T_r \\ -\mathbf{v}_y/T_r \end{bmatrix}.$$
(6)

Since **D** is a symmetric positive definite tensor, consider an unit vector $\mathbf{n} = (n_x \quad n_y)^T$ at an arbitrary direction, one would have

$$\nu = \mathbf{n}^T \mathbf{D} \mathbf{n} = \mathcal{D}_{xx} n_x^2 + 2\mathcal{D}_{xy} n_x n_y + \mathcal{D}_{yy} n_y^2 > 0.$$
(7)

Consider the Jacobian of the flux projected along **n**,

$$\mathbf{A}_{n} = \frac{\partial(\mathbf{F} \cdot \mathbf{n})}{\partial \mathbf{U}} = \frac{\partial(\mathbf{F}_{x}n_{x} + \mathbf{F}_{y}n_{y})}{\partial \mathbf{U}} = \begin{bmatrix} 0 & -\mathbf{D}_{xx}n_{x} - \mathbf{D}_{xy}n_{y} & -\mathbf{D}_{xy}n_{x} - \mathbf{D}_{yy}n_{y} \\ -n_{x}/T_{r} & 0 & 0 \\ -n_{y}/T_{r} & 0 & 0 \end{bmatrix}.$$
 (8)

It has the following eigenvalues

$$\lambda_1 = -\sqrt{\frac{\nu}{T_r}}, \quad \lambda_2 = -\sqrt{\frac{\nu}{T_r}}, \quad \lambda_3 = 0.$$
(9)

The first two nonzero eigenvalues indicate that the system describes a wave propagating isotropically. The third eigenvalue corresponds to the inconsistency damping mode.³⁵ The relaxation time T_r does not affect

the steady solution, and thus can be defined solely for the purpose of accelerating the convergence to the steady state. For simplicity, T_r is defined as

$$T_r = L_r^2, \quad L_r = \frac{1}{2\pi},$$
 (10)

where the length scale L_r , according to Nishikawa's work,^{35–39} has been defined to maximize the effect of propagation. The absolute Jacobian $|\mathbf{A}_n|$ is constructed by the right-eigenvector matrix \mathbf{R}_n , and the diagonal eigenvalue-matrix Λ_n ,

$$\mathbf{R}_{n} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ n_{x}/\sqrt{\nu T_{r}} & n_{x}/\sqrt{\nu T_{r}} & -2(\mathbf{D}_{xx}n_{x} + \mathbf{D}_{xy}n_{y}) \\ n_{y}/\sqrt{\nu T_{r}} & n_{y}/\sqrt{\nu T_{r}} & -2(\mathbf{D}_{xy}n_{x} + \mathbf{D}_{yy}n_{y}) \end{bmatrix}, \Lambda_{n} = \begin{bmatrix} -\lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
(11)

where $\lambda = |\lambda_1| = |\lambda_2| = \sqrt{\nu/T_r}$. Hence,

$$\mathbf{A}_{n}| = \mathbf{R}_{n}|\Lambda_{n}|\mathbf{R}_{n}^{-1} = \lambda \begin{bmatrix} 1 & 0 & 0\\ 0 & n_{x}(\mathbf{D}_{xx}n_{x} + \mathbf{D}_{xy}n_{y}) & n_{y}(\mathbf{D}_{xx}n_{x} + \mathbf{D}_{xy}n_{y})\\ 0 & n_{x}(\mathbf{D}_{xy}n_{x} + \mathbf{D}_{yy}n_{y}) & n_{y}(\mathbf{D}_{xy}n_{x} + \mathbf{D}_{yy}n_{y}) \end{bmatrix}.$$
 (12)

III. Reconstruction Discontinuous Galerkin Methods

Discontinuous Galerkin methods are a family of numerical method that combines the advantages of classic finite volume methods and finite element methods. With the method of lines, it would expressed the solution as

$$\mathbf{u}_{h} = \sum_{j=1}^{N} u_{j}(t) B_{j}(x, y), \tag{13}$$

where B_j is basis function in the broke Sobolev space. Multiply test function, chosen as the basis function B_i , to Eq. (5), and integrate by parts would yield

$$\frac{d}{d\tau} \int_{\Omega_e} \mathbf{U}_h B_i d\Omega + \oint_{\partial\Omega_e} \mathbf{F} \cdot \mathbf{n} d\Gamma - \int_{\Omega_e} \mathbf{F} \cdot \nabla B_i d\Omega = \int_{\Omega_e} \mathbf{S} B_i d\Omega, \quad i = 1, \cdots, N.$$
(14)

In short, we could express it as the semi-discretized form

$$\mathbf{M}\frac{d\mathbf{U}}{d\tau} = \mathbf{R},\tag{15}$$

where \mathbf{M} is the mass matrix, whose entries are

$$m_{ij} = \int_{\Omega_e} B_i B_j d\Omega, \quad i, j = 1, 2, \cdots, N,$$
(16)

and \mathbf{R} is the residual vector, defined as

$$r_{i} = \int_{\Omega_{e}} \left(\mathbf{F}_{x} \frac{\partial B_{i}}{\partial x} + \mathbf{F}_{y} \frac{\partial B_{i}}{\partial y} + \mathbf{S}B_{i} \right) d\Omega - \oint_{\partial\Omega_{e}} \mathbf{F}_{n}B_{i}d\Gamma, \quad i = 1, 2, \cdots, N.$$
(17)

In the traditional nodal discontinues Galerkin methods, numerical polynomial solutions in each element are expressed using either standard Lagrange finite element or hierarchical node-based basis function. In the implementation of the DGMs in this paper, however, modal DG is adopted. The numerical polynomial solutions are represented using a Taylor series expansion at the cell center and normalized to improve the conditioning of the system matrix. For example, the quadratic polynomial solutions could be expressed as follows,

$$\begin{aligned} \mathbf{U}_{h} = \mathbf{U}_{c} + \frac{\partial \mathbf{U}}{\partial x} \Big|_{c} (x - x_{c}) + \frac{\partial \mathbf{U}}{\partial y} \Big|_{c} (y - y_{c}) \\ + \frac{\partial^{2} \mathbf{U}}{\partial x^{2}} \Big|_{c} \frac{(x - x_{c})^{2}}{2} + \frac{\partial^{2} \mathbf{U}}{\partial y^{2}} \Big|_{c} \frac{(y - y_{c})^{2}}{2} + \frac{\partial^{2} \mathbf{U}}{\partial x \partial y} \Big|_{c} (x - x_{c})(y - y_{c}), \end{aligned}$$
(18)

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which could be further expressed as cell-averaged values, i.e., $\tilde{\mathbf{U}}$, and the derivatives at the center.

$$\mathbf{U}_{h} = \widetilde{\mathbf{U}}B_{1} + \frac{\partial \mathbf{U}}{\partial x}\Big|_{c} \Delta x B_{2} + \frac{\partial \mathbf{U}}{\partial y}\Big|_{c} \Delta y B_{3} + \frac{\partial^{2} \mathbf{U}}{\partial x^{2}}\Big|_{c} \Delta x^{2} B_{4} + \frac{\partial^{2} \mathbf{U}}{\partial y^{2}}\Big|_{c} \Delta y^{2} B_{5} + \frac{\partial^{2} \mathbf{U}}{\partial x \partial y}\Big|_{c} \Delta x \Delta y B_{6},$$
(19)

where $\Delta x = 0.5(x_{\text{max}} - x_{\text{min}})$ and $\Delta y = 0.5(y_{\text{max}} - y_{\text{min}})$. And x_{max} , x_{min} , y_{max} , and y_{min} are the maxium and minimum coordinates in the cell Ω_e in x-, and y- directions respectively. The basis functions are given as follows

$$B_{1} = 1, B_{2} = \frac{x - x_{c}}{\Delta x}, B_{3} = \frac{y - y_{c}}{\Delta y}, B_{4} = \frac{1}{2} \left(B_{2}^{2} - \int_{\Omega_{e}} B_{2}^{2} d\Omega \right),$$

$$B_{5} = \frac{1}{2} \left(B_{3}^{2} - \int_{\Omega_{e}} B_{3}^{2} d\Omega \right), B_{6} = B_{2}B_{3} - \int_{\Omega_{3}} B_{2}B_{3} d\Omega.$$
(20)

Compared with reconstructed FV methods, the DGM would require more degrees of freedom, additional domain integration, and more Gauss quadrature points for the boundary integration, which leads to more computational costs and storage requirements. Inspired by the DGM from Dumbser et al. in the frame of PnPm scheme,¹²⁻¹⁴ a hierarchical WENO-based rDG method^{30,32} is designed to achieve high order of accuracy while reducing the computational cost. As a matter of fact, rDG method could provide a unified formulation for both FV and DG methods. The standard FV and DG method would be nothing but special cases in rDG frame work, and thus allow for a direct efficiency comparison.

Based on different DGMs, some effective discretization hyperbolic rDG methods will be presented to deal with the derived first-order hyperbolic system (FOHS). The format $\mathbf{A} + \mathbf{B}$ is used to indicate the discretization method for the system, where A refers to the discretization method for φ and B refers to the discretization method for its derivatives. Note that what make the hyperbolic DG methods based on FOHS different from standard DGMs is that the system is in the partial differential equation (PDE) level other than in the discretization level. While the choice of **B** would be either standard DG(Pn) method or rDG(PnPm) method, we have one more option in A. Since FOHS would introduce the derivatives into the system as auxiliary variables, one can construct higher order polynomial in the primary variable while keeping the minimal degrees of freedom. In FV methods, this technique corresponds to Scheme-II in Ref.,³⁷ which replaces the LSQ gradients in the primary variable reconstruction by the auxiliary variables. It thus eliminates the need for the LSQ gradient computation for the primary variable, but the degrees of freedom remains the same. In the DGMs, it eliminates the need to store and solve for high-order moments for the primary variable,³³ and effectively reducing the degrees of freedom and the number of discrete equations. This new approach, termed DG(P0Pk), can deliver high order accuracy in φ with the minimal computational cost and storage requirement, where k is the order of polynomial we construct for φ with the information from its derivatives.

In this work, Gaussian quadrature formulas are used to compute the integration. We tabulate the the degrees of freedom and optimal order of accuracy in φ and its derivatives, as well as number of minimal needed Gauss quadrature points for face and domain integration in Table 1. The top three schemes are straightforward DG discretizations of FOHS, which achieve first, second, and third-order accuracy for all variables. The next three schemes correspond to their efficient variants: the first one constructs a linear polynomial of φ by using $(\mathbf{v}_x, \mathbf{v}_y)$ as its slopes; the second and third ones replace higher-order moments of φ by using the information of (v_x, v_y) . The next two schemes are rDG constructions: the first one corresponds to a second-order finite-volume scheme, and the second one is a popular third-order rDG scheme. The next four schemes, DG+rDG, are more efficient rDG constructions, whereonly one degree of freedom is retained for φ for all accuracy orders. Finally, the BR2 schemes are listed for comparison. As can be seen, the DG+rDG schemes achieve higher-order accuracy in either φ or (v_x, v_y) , or both for comparable numbers of degrees of freedom. It is possible to match the degrees of freedoms of DG(P0P2)+rDG(P1P2) (and even DG(P0P2)+rDG(P1P2)) and BR2(P2) by unifying the cross derivative terms of (v_x, v_y) as in Ref.³³ But such is left as future work as these schemes are already more efficient than BR2(P2), which can be seen also in terms of the number of quadrature points. It is clearly seen that hyperbolic rDG methods could achieve high order of accuracy while using less degrees of freedom and also less Gauss quadrature points compared with standard DGMs. Finally, it should be noted that BR2 schemes require the same order of polynomials and their storage for the solution and the diffusive fluxes, and therefore the hyperbolic DG/rDG methods and

the BR2 method (or any FOS-based methods) has the same storage requirement. The hyperbolic DG/rDG methods solve for the solution and the derivatives in a coupled manner, thus resulting in a larger system of discrete equations; the BR2 schemes use the FOS formulation in a decoupled manner by explicitly computing the diffusive fluxes and then substituting them in the primal discretization.

Table 1: Comparison between different hyperbolic rDG methods and the BR2 method in terms of degree	es
of freedom, optimal order and number of Gaussian quadrature points in 2D. In BR2, the degrees of freedo	m
for the diffusive fluxes are not included.	

Schomo	Degrees of freedom	Observed order		Number of Gauss points	
Scheme		φ	$\mathrm{v}_x,\mathrm{v}_y$	Face integral	domain integral
DG(P0)+DG(P0)	3	1	1	1	1
DG(P1)+DG(P1)	9	2	2	2	3
DG(P2)+DG(P2)	18	3	3	3	6
DG(P0P1)+DG(P0)	3	2	1	2	1
DG(P0P1)+DG(P1)	7	2	2	2	3
DG(P0P2)+DG(P1)	7	3	2	2	3
rDG(P0P1)+rDG(P0P1)	3	2	2	2	1
rDG(P1P2)+rDG(P1P2)	9	3	3	3	4
DG(P0P1)+rDG(P0P1)	3	2	2	2	1
DG(P0P2)+rDG(P0P1)	3	3	2	2	1
DG(P0P2)+rDG(P1P2)	7	3	3	2	4
DG(P0P3)+rDG(P1P2)	7	4	3	3	4
DG(P1)-BR2	3	2	1	2	3
DG(P2)-BR2	6	3	2	3	6

The hyperbolic formulation in FOHS would allow hyperbolic DGMs to use well-established methods for hyperbolic system. Instead of using BR2, DDG or other usual diffusion scheme for DGMs, the numerical flux is computed by the upwind flux as

$$\mathbf{F}_{ij} = \frac{1}{2} (\mathbf{F}_L + \mathbf{F}_R) \cdot \mathbf{n}_{ij} - \frac{1}{2} |\mathbf{A}_n| (\mathbf{U}_R - \mathbf{U}_L), \qquad (21)$$

where the subscripts L and R indicate the values evaluated at the face by the polynomials in the elements i and j, respectively, and \mathbf{n}_{ij} denotes a unit face-normal vector pointing from the element i to the neighbor element j. In contrast, the Rusanov (or Local-Lax-Friedrich) flux as employed in Ref.³³ is given by

$$\mathbf{F}_{ij} = \frac{1}{2} (\mathbf{F}_L + \mathbf{F}_R) \cdot \mathbf{n}_{ij} - \frac{\lambda}{2} (\mathbf{U}_R - \mathbf{U}_L), \qquad (22)$$

which is much more dissipative than the upwind flux and yields lower levels of errors as we will show later.

Boundary conditions are enforced weakly through the numerical flux in a similar manner as described for one-dimensional hyperbolic schemes in Ref.³⁹ For all test problems, the Dirichlet condition is considered, and therefore the solution φ is given on boundaries. At a boundary face, \mathbf{n}_{ij} is taken to be outward, and thus \mathbf{U}_R is considered as a boundary state. The boundary condition is incorporated into the boundary state as

$$\mathbf{U}_R = (\varphi_b, \mathbf{v}_n n_x + \partial_s \varphi_b t_x, \mathbf{v}_n n_y + \partial_s \varphi_b t_y), \tag{23}$$

where φ_b is the value given as a boundary condition, and $\partial_s \varphi_b$ is the tangential derivative that can be obtained from the given boundary condition, \mathbf{v}_n is the face-normal projection of $(\mathbf{v}_x, \mathbf{v}_y)$ evaluated at the left (interior) state \mathbf{U}_L , $\mathbf{n}_{ij} = (n_x, n_y)$, and (t_x, t_y) denotes a unit tangent vector of the boundary face. Note that s is taken to be positive in the counterclockwise direction along a boundary, and the tangent vector is also taken in the same direction. In the case of a unit square domain, the boundary state becomes

$$\mathbf{U}_R = (\varphi_b, \mathbf{v}_x, \partial_y \varphi_b),\tag{24}$$

$$\mathbf{U}_R = (\varphi_b, \partial_x \varphi_b, \mathbf{v}_y),\tag{25}$$

at the top and bottom boundaries.

Note that the normal component v_n may be specified in place of φ_b in the case of Neumann problems. As discussed in,³⁵ the hyperbolic diffusion system has one wave going out of the domain, and therefore one quantity should be left unspecified, which corresponds to the normal derivative v_n in the Dirichlet case (or φ in the Neumann case). Or it may be argued that since the hyperbolic diffusion system is equivalent to the original diffusion equation in the pseudo steady state, the boundary condition should also be the same as the original problem. The tangential derivative can be specified since φ is known in the Dirichlet case, but it is not necessary; the results are very similar with and without specifying $\partial_s \varphi_b$.

To advance the solution in time, explicit three-stage TVD Runge-Kutta (TVDRK3) scheme and implicit BDF1 scheme have been employed in this work. To accelerate the convergence, the local time step is employed and determined as

$$\Delta t = \text{CFL} \frac{2\Omega_e}{\sum_j (\nu/L_r A_{ij} + \Omega_j/T_r)}.$$
(26)

For TVDRK3 time marching scheme, hyperbolic rDG method typically would require $CFL \leq 1.5$ to ensure stability. However, for implicit BDF1 method, one can set CFL as infinity (CFL= 10^{15}) for all the case. In other words, the pseudo time term could be dropped for implicit time marching. For implicit scheme, GMRES+LU-SGS^{21,43} has been implemented to solve the linear system, where LU-SGS serves as the preconditioner.

IV. Numerical Examples

Several steady model diffusion problems in a unit square are considered in this section. All the cases would be computed on three types of grids. The first and second types are regular and irregular triangular grids with 9×9 , 17×17 , 33×33 , and 65×65 nodes. The irregular grid is generated from the regular grid by random diagonal swapping and nodal perturbation. The third type is a set of heterogeneous grids with 12×11 , 23×21 , 45×41 , and 99×81 nodes. The second grid of every type is shown in Figure 1.

In our work, the quality of these 3 types of grids is investigated as follows. For any triangle, there exists a ratio,

$$r = \frac{R_{out}}{R_{in}},\tag{27}$$

where, R_{out} denotes the radius of the circumscribed circle, and R_{in} refers to the radius of the inscribed circle. The ratio for the equilateral triangle, the best choice for numerical computation, is equal to 2.0. Therefore, in this work, the quality of meshes would be measured through

$$R = \frac{r-2}{2}.$$
(28)



Figure 1: The second mesh of every type, that is, 17×17 regular grid, 17×17 irregular grid, and 23×21 heterogeneous grid.

Clearly, larger R indicates that the triangle meshes deviates greatly from the equilateral triangle. While the first type of mesh, i.e., the regular mesh, would remain same R for the entire computation domain, others may encounter large R. The contours of R for the coarsest mesh from the irregular and heterogeneous mesh are shown in Figure 2. It could be observed that, for the first heterogeneous mesh, maximum of reached 80, which are almost 2 orders of magnitude higher than the one of the irregular mesh. This demonstrates the bad quality of the heterogeneous mesh, and also will demonstrate the robustness and accuracy of developed methods.



Figure 2: Distribution of R in the coarsest irregular mesh (left) and heterogeneous mesh (right).

In this paper, all the cases are computed with the initial solution set as 1.0 everywhere and the steady state is considered to be reached when the residual in φ drops below 10^{-12} in the L_2 norm.

A. Case I, Scalar coefficient without source term

To begin with, a steady diffusion problem in a unit square is considered with the exact solution given by

$$\varphi(x,y) = \frac{\sinh(\pi x)\sin(\pi y)\sinh(\pi y)\sin(\pi x)}{\sinh(\pi)},\tag{29}$$

and

$$\mathbf{D} = \mathbf{I}, \quad f = 0, \tag{30}$$

where **I** is the 2×2 identity matrix.

The grid refinement study is carried out for regular, irregular and heterogeneous mesh to access the accuracy and the robustness of the presented hyperbolic rDG methods using BDF1. L_2 norm of the difference between the numerical results and the exact ones is used as the error measurement. All norms are computed by Gaussian quadrature. The grid refinement study results for regular, irregular and heterogeneous meshes are shown in Figure 3.

In all cases, the same order of accuracy, at least, has been achieved as expected. Note that the same order of accuracy is expected even when a higher-order polynomial is used in the primary variable; this is consistent with results for other hyperbolic schemes.^{37,39} Exceptions are DG(P0P1)+DG(P0), DG(P0P2)+DG(P1), and DG(P0P3)+rDG(P1P2), which achieve one-order higher accuracy in the primary variable. Such results have never been observed in the hyperbolic method before. Note, however, that DG(P0P3)+rDG(P1P2) is not stable on the heterogeneous mesh. Therefore, WENO reconstruction has to be used to guarantee the stability of such scheme with the cost of achieving only 3rd order of accuracy for variable φ . Note that though upwind flux is implemented for all hyperbolic DGMs here, we use Rusanov flux scheme with DG(P0P3)+rDG(P1P2) in irregular grids to compare. Clearly, Rusanov flux scheme would lead to less accurate results than upwind method due to too much dissipation, especially in φ .

Next, the comparison between hyperbolic DGMs and BR2 is carried out on regular, irregular and heterogeneous meshes using explicit TVDRK3, and the order of accuracy results are shown in Figure 4. It is obvious that, DG(P1)-BR2 could achieve 2nd order of accuracy for variable , and 1st order of accuracy for gradient of variable. Likewise, DG(P2)-BR2 could achieve 3rd order of accuracy for variable φ , and 2nd order of accuracy for gradients. Note that the gradients are obtained always with one-order-lower accuracy than the primary variable, which is typical in conventional DGMs.



Figure 3: Grid refinement study of Case I on regular grids(left), irregular grids (middle) and heterogeneous grids(right) using implicit BDF1.



Figure 4: Comparison between hyperbolic rDG methods and BR2 scheme in terms of order of accuracy on regular grids(left), irregular grids (middle) and heterogeneous grids(right) for Case I using explicit TVDRK3.



Figure 5: Comparison between hyperbolic rDG methods and BR2 scheme in terms of CPU time on 17×17 regular grids(left), 17×17 irregular grids (middle) and 23×21 heterogeneous grids(right) for Case I using explicit TVDRK3.

Several hyperbolic DGMs are presented to compare with BR2 scheme. Note that for heterogeneous grids, DG(P0P3) + rDG(P1P2) is carried out with WENO reconstruction to remain stability. Results show that, our method is comparable to the BR2 method in terms of computational cost and accuracy. Note in particular that the same order of accuracy has been achieved by DG(P1)-BR2 and DG(P0P1) + DG(P0), both of which involve exactly three discrete equations, and by DG(P2)-BR2 and DG(P0P2) + DG(P1), which involve six and seven discrete equations, respectively.



Figure 6: Time steps and CPU time required to reach the steady state for regular grids using explicit TVDRK3.

Moreover, hyperbolic DG would also outperform BR2 in terms of the time steps and CPU time required to reach convergence. The simulation is carried out by hyperbolic DG method and BR2 method using TVDRK3 on the second mesh of each type, with the results shown in Figure 5. Clearly, hyperbolic DGMs would be way less time consuming than standard BR2 scheme. As has been demonstrated for diffusion problems with other hyperbolic discretization methods,^{35,37,39} the hyperbolic method is known to achieve iterative convergence acceleration by the elimination of numerical stiffness due to second-derivative diffusion

operators.

To better illustrate the high efficiency of the developed hyperbolic rDG methods, Figure 6 shows the required total time steps and CPU time for hyperbolic rDG methods and traditional BR2 methods to reach the convergence on all each regular mesh while keeping the CFL number when refining the mesh. As expected, $O(h^2)$ time steps would be required for traditional diffusion scheme while the total iterations increase only linearly with the all the hyperbolic rDG methods. In other words, the presented methods lead to O(1/h) acceleration in the steady convergence over traditional methods.

B. Case II, Scalar coefficient with source term

A steady diffusion problem with source term is considered in this case. The exact solution is given by

$$\varphi(x,y) = 2\cos(\pi x)\sin(2\pi y) + 2, \tag{31}$$

and

$$\mathbf{D} = \mathbf{I}, \quad f = 10\pi^2 \cos(\pi x) \sin(2\pi y). \tag{32}$$

Similarly, the simulations are performed in those three types of grids using hyperbolic DGMs with implicit BDF1. The numerical results are shown in Figure 7.

In this case, the property of one-order higher accuracy in the primary variable for DG(P0P1)+DG(P0)and DG(P0P2)+DG(P1) is lost; but DG(P0P3)+rDG(P1P2) still gives fourth-order accuracy in φ and third-order accuracy in the derivatives. Otherwise, the same order of accuracy is successfully achieved for all variables as expected.



Figure 7: Grid refinement study of Case II on regular grids(left), irregular grids (middle) and heterogeneous grids(right) using implicit BDF1.

C. Case III, Tensor coefficient with source term

To test the developed methods further, the following model diffusion problem is tested, with the exact solution given by

$$\varphi(x,y) = 1 - \tanh\left(\frac{(x-0.5)^2 + (y-0.5)^2}{0.01}\right),\tag{33}$$

and the diffusion tensor \mathbf{D} is given by

$$\mathbf{D} = \begin{bmatrix} (x+1)^2 + y^2 & -xy \\ -xy & (y+1)^2 \end{bmatrix}.$$
(34)

The source term can be computed as

$$f = -\nabla \cdot (\mathbf{D}\nabla\varphi) = 100(1 - A^2)(200AB + C), \tag{35}$$

where

$$A = \tanh\left(\frac{(x-0.5)^2 + (y-0.5)^2}{0.01}\right),\tag{36}$$

$$B = -4x^{3}(x+1) + (4y^{2} - 4y + 3)x^{2} + 2(y+1)x - 4y^{3}(y+1) + 2(y^{2} + y - 1),$$
(37)

and

$$C = x(4x+7) + y(6y+7).$$
(38)

Again, grid refinement study is carried out for all three types of meshes using implicit BDF1, with the results shown in Figure 8. Similar to the second case, DG(P0P1)+DG(P0) and DG(P0P2)+DG(P1) would not have super-convergence in φ while all other hyperbolic DGMs delivering designed order of accuracy for both variables and their derivatives, indicating the presented scheme is robust and can provide an attractive alternatives for using DGMs for diffusion problems.



Figure 8: Grid refinement study of Case III on regular grids(left), irregular grids (middle) and heterogeneous grids(right) using implicit BDF1.

V. Concluding Remarks

Reconstructed discontinuous Galerkin methods have been developed for solving diffusion equations based on a first-order hyperbolic system formulation. By judiciously using a Taylor basis in the DG formulation and solution derivatives handily available in the FOHS formulation, the resultant hyperbolic rDG methods have the same number of degrees of freedom as the standard DG methods. A number of test cases for different diffusion equations have been presented to assess accuracy and performance of the newly developed hyperbolic rDG methods in comparison with the standard BR2 DG method. Numerical experiments demonstrate that the hyperbolic rDG methods are able to achieve the designed optimal order of accuracy for both solutions and their derivatives on regular, irregular, and heterogeneous girds, and outperform the BR2 method in terms of the magnitude of the error, the order of accuracy, the size of time steps, and the CPU times required to achieve steady state solutions, indicating that the developed hyperbolic rDG methods provide an attractive and probably an even superior alternative for solving the diffusion equations. Our future work is focused to extend the developed hyperbolic rDG methods for solving the advection-diffusion equations, and the incompressible and compressible Navier-Stokes equations on fully irregular, adaptive, anisotropic, unstructured grids.

Acknowledgments

This work has been funded by the U.S. Army Research Office under the contract/grant number W911NF-16-1-0108 with Dr. Matthew Munson as the program manager.

References

¹D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM journal on numerical analysis*, 39(5):1749–1779, 2002.

²H. L. Atkins and C. W. Shu. Quadrature Free Implementation of Discontinuous Galerkin Method for Hyperbolic Equations. *AIAA Journal*, 36:775–782, 1998.

³F. Bassi and S. Rebay. A High-Order Accurate Discontinuous Finite Element Method for the Numerical Solution of the Compressible Navier-Stokes Equations. *Journal of Computational Physics*, 131(2):267–279, 1997.

⁴F. Bassi and S. Rebay. High-Order Accurate Discontinuous Finite Element Solution of the 2D Euler Equations. *Journal of Computational Physics*, 138:251–285, 1997.

⁵F. Bassi and S. Rebay. GMRES discontinuous Galerkin solution of the Compressible Navier-Stokes Equations. Discontinuous Galerkin Methods, Theory, Computation, and Applications. Edited by B. Cockburn, G. E. Karniadakis, and C. W. Shu. Lecture Notes in Computational Science and Engineering, 11:197–208, 2000.

⁶F. Bassi and S. Rebay. Discontinuous Galerkin Solution of the Reynolds-Averaged Navier-Stokes and κ -ω Turbulence Model Equations. Computers & Fluids, 34(4-5):507–540, 2005.

⁷C. E. Baumann and J. T. Oden. A Discontinuous *hp* Finite Element Method for the Euler and Navier-Stokes Equations. International Journal for Numerical Methods in Fluids, 31(1):79–95, 1999.

⁸B. Cockburn, S. Hou, and C. W. Shu. TVB Runge-Kutta Local Projection Discontinuous Galerkin Finite Element Method for conservation laws IV: the Multidimensional Case. *Journal of Mathematical Physics*, 55:545–581, 1990.

⁹B. Cockburn, G. Karniadakis, and Shu C. W. The Development of Discontinuous Galerkin Method. Discontinuous Galerkin Methods, Theory, Computation, and Applications. Edited by B. Cockburn, G. E. Karniadakis, and C. W. Shu. Lecture Notes in Computational Science and Engineering, 11:5–50, 2000.

¹⁰B. Cockburn and C. W. Shu. The Runge-Kutta Discontinuous Galerkin Method for conservation laws V: Multidimensional System. *Journal of Computational Physics*, 141:199–224, 1998.

¹¹B. Cockburn and C.W. Shu. The Local Discontinuous Galerkin Method for Time-dependent Convection-Diffusion System. SIAM, Journal of Numerical Analysis, 35(6):2440–2463, 1998.

¹²M. Dumbser. Arbitrary High Order PNPM Schemes on Unstructured Meshes for the Compressible Navier-Stokes Equations. Computers & Fluids, 39(1):60–76, 2010.

¹³M. Dumbser and O. Zanotti. A unified framework for the construction of one-step finite volume and discontinuous Galerkin schemes on unstructured meshes. *Journal of Computational Physics*, 227(18):8209–8253, 2008.

 $^{14}\mathrm{M}.$ Dumbser and O. Zanotti. Very high order $\mathrm{P_NP_M}$ schemes on unstructured meshes for the resistive relativistic MHD equations. Journal of Computational Physics, 228(18):6991–7006, 2009.

¹⁵Bassi .F, Rebay S., Mariotti G., Pedinotti S., and Savini M. High-Order Accurate Discontinuous Finite Element Method for Inviscid and Viscous Turbomachinery Flows. in: Decuypere R, Dibelius G, editors. 2nd European Conference on Turbomachinery Fluid Dynamics and Thermodynamics, Technologisch Instituut, Antwerpen, Belgium, pages 99–108, 1997.

¹⁶K. J. Fidkowski, T. A. Oliver, J. Lu, and D. L. Darmofal. *p*-Multigrid solution of high-order discontinuous Galerkin discretizations of the compressible NavierStokes equations. *Journal of Computational Physics*, 207(1):92–113, 2005.

¹⁷G. Gassner, F. Lorcher, and C. D. Munz. A contribution to the construction of diffusion fluxes for finite volume and discontinuous Galerkin schemes. *Journal of Computational Physics*, 224(2):1049–1063, 2007.

¹⁸D. Helenbrook, B. T. Mavriplis and H. L. Atkins. Analysis of p-Multigrid for Continuous and Discontinuous Finite Element Discretizations. AIAA Paper, 2003–3989, 2003.

¹⁹J. S. Hesthaven and T. Warburton. Nodal Discontinuous Galerkin Methods: Algorithms, Analysis, and Applications. Springer, 2008.

²⁰Y. Liu and H. Nishikawa. Third-order inviscid and second-order hyperbolic Navier-Stokes solvers for three-dimensional inviscid and viscous flows. In 64th AIAA Fluid Dynamics Conference, AIAA Paper 2016-3969, Washington, D.C., 2016.

²¹H. Luo, J. D. Baum, and R. Löhner. A Fast, Matrix-free Implicit Method for compressible flows on Unstructured Grids. Journal of Computational Physics, 146(2):664–690, 1998.

²²H. Luo, J. D. Baum, and R. Löhner. A *p*-Multigrid Discontinuous Galerkin Method for the Euler Equations on Unstructured Grids. *Journal of Computational Physics*, 211(2):767–783, 2006. ²³H. Luo, J. D. Baum, and R. Löhner. A Hermite WENO-based Limiter for Discontinuous Galerkin Method on Unstructured Grids. *Journal of Computational Physics*, 225(1):686–713, 2007.

²⁴H. Luo, J. D. Baum, and R. Löhner. A Discontinuous Galerkin Method Using Taylor Basis for Compressible Flows on Arbitrary Grids. *Journal of Computational Physics*, 227(20):8875–8893, 2008.

²⁵H. Luo, J. D. Baum, and R. Löhner. A Fast, p-Multigrid Discontinuous Galerkin Method for Compressible Flows at All Speeds. AIAA Journal, 46(3):635–652, 2008.

²⁶H. Luo, J. D. Baum, and R. Löhner. On the Computation of Steady-State Compressible Flows Using a Discontinuous Galerkin Method. *International Journal for Numerical Methods in Engineering*, 73(5):597–623, 2008.

²⁷H. Luo, L. Luo, A Ali, R. Nourgaliev, and C. Cai. A Parallel, Reconstructed Discontinuous Galerkin Method for the Compressible Flows on Arbitrary Grids. *Communication in Computational Physics*, 9(2):363–389, 2011.

²⁸H. Luo, L. Luo, and R. Nourgaliev. A reconstructed discontinuous Galerkin method for the Euler equations on arbitrary grids. Communications in Computational Physics, 12(5):1495–1519, 2012.

²⁹H. Luo, L. Luo, R. Nourgaliev, V. Mousseau, and N. Dinh. A Reconstructed Discontinuous Galerkin Method for the Compressible Navier-Stokes Equations on Arbitrary Grids. *Journal of Computational Physics*, 229(19):6961–6978, 2010.

³⁰H. Luo, Y. Xia, S. Li, and R. Nourgaliev. A Hermite WENO Reconstruction-Based Discontinuous Galerkin Method for the Euler Equations on Tetrahedral grids. *Journal of Computational Physics*, 231(16):5489–5503, 2012.

³¹H. Luo, Y. Xia, and R. Nourgaliev. A Class of Reconstructed Discontinuous Galerkin Methods in Computational Fluid Dynamics. International Conference on Mathematics and Computational Methods Applied to Nuclear Science and Engineering (M&C2011), Brazil, 2011.

³²H. Luo, Y. Xia, S. Spiegel, R. Nourgaliev, and Z. Jiang. A reconstructed discontinuous Galerkin method based on a hierarchical WENO reconstruction for compressible flows on tetrahedral grids. *Journal of Computational Physics*, 236:477–492, 2013.

³³A. Mazaheri and H. Nishikawa. Efficient high-order discontinuous Galerkin schemes with first-order hyperbolic advectiondiffusion system approach. *Journal of Computational Physics*, 2016.

³⁴Y. Nakashima, N. Watanabe, and H. Nishikawa. Hyperbolic Navier-Stokes solver for three-dimensional flows. In 54th AIAA Aerospace Sciences Meeting, AIAA Paper 2016-1101, San Diego, CA, 2016.

³⁵H. Nishikawa. A first-order system approach for diffusion equation. I: Second-order residual-distribution schemes. *Journal of Computational Physics*, 227(1):315–352, 2007.

³⁶H. Nishikawa. A first-order system approach for diffusion equation. II: Unification of advection and diffusion. *Journal of Computational Physics*, 229(11):3989–4016, 2010.

³⁷H. Nishikawa. First-, second-, and third-order finite-volume schemes for diffusion. *Journal of Computational Physics*, 256:791–805, 2014.

³⁸H. Nishikawa. First, second, and third order finite-volume schemes for advection–diffusion. Journal of Computational Physics, 273:287–309, 2014.

³⁹H. Nishikawa and Philip L. Roe. Third-order active-flux scheme for advection diffusion: Hyperbolic diffusion, boundary condition, and Newton solver. *Computers and Fluids*, 125:71–81, February 2016.

⁴⁰J. Peraire and P. O. Persson. The Compact Discontinuous Galerkin Method for Elliptic Problems. SIAM Journal on Scientific Computing, 30(4):1806–1824, 2008.

⁴¹P. Rasetarinera and M. Y. Hussaini. An Efficient Implicit Discontinuous Spectral Galerkin Method. *Journal of Computational Physics*, 172:718–738, 2001.

⁴²W.H. Reed and T.R. Hill. Triangular Mesh Methods for the Neutron Transport Equation. Los Alamos Scientific Laboratory Report, LA-UR-73-479, 1973.

⁴³Dmitri Sharov, Hong Luo, Joseph D Baum, and Rainald Löhner. Implementation of unstructured grid gmres+ lu-sgs method on shared-memory, cache-based parallel computers. *AIAA paper*, 927:2000, 2000.

⁴⁴Z. J. Wang, K. Fidkowski, R. Abgrall, F. Bassi, D. Caraeni, A. Cary, H. Deconinck, R. Hartmann, K. Hillewaert, H. T. Huynh, et al. High-order CFD methods: current status and perspective. *International Journal for Numerical Methods in Fluids*, 2013.

⁴⁵T. C. Warburton and G. E. Karniadakis. A Discontinuous Galerkin Method for the Viscous MHD Equations. *Journal of Computational Physics*, 152:608–641, 1999.

⁴⁶L. Zhang, L. Wei, L. He, X. Deng, and H. Zhang. A class of hybrid DG/FV methods for conservation laws I: Basic formulation and one-dimensional systems. *Journal of Computational Physics*, 231(4):1081–1103, 2012.

⁴⁷Laiping Zhang, Liu Wei, He Lixin, Deng Xiaogang, and Zhang Hanxin. A class of hybrid dg/fv methods for conservation laws ii: Two-dimensional cases. *Journal of Computational Physics*, 231(4):1104–1120, 2012.