# Reconstructed Discontinuous Galerkin Methods Based on First-Order Hyperbolic System for Advection-Diffusion Equations 

Jialin Lou * , Lingquan Li ${ }^{\dagger}$, Xiaodong Liu ${ }^{\ddagger}$ and Hong Luo ${ }^{\S}$<br>North Carolina State University, Raleigh, NC 27695, United States<br>and<br>Hiroaki Nishikawa ${ }^{\top}$<br>National Institute of Aerospace, Hampton, VA 23666, United States

In this study, reconstructed Discontinuous Galerkin (rDG) methods are developed for solving advection-diffusion equations based on a first-order hyperbolic system (FOHS) formulation. The developed hyperbolic rDG methods are reliable, accurate, efficient, and robust by combining the advantages of both FOHS and rDG methods. The presented methods have the same number of degrees-of-freedom as the conventional DG methods. Both hybrid least-squares reconstruction and variational reconstruction has been implemented in the study to deliver high order numerical solution while keeping the total degrees of freedom relatively small. A number of advection-diffusion test cases with a wide range of Reynolds numbers, including boundary layer type problems are presented to assess accuracy and performance of the newly developed hyperbolic rDG methods. Numerical experiments demonstrate that the hyperbolic rDG methods are able to achieve the designed optimal order of accuracy for both solutions and their derivatives on regular, irregular, and heterogeneous girds, indicating that the developed hyperbolic rDG methods provide an attractive and probably an even superior alternative for solving the advection-diffusion equations.

## I. Introduction

Nowadays, the discontinuous Galerkin (DG) methods, originally developed for solving the neutron transport, ${ }^{37}$ have shown increasing attention in science and engineering filed for solving conservation laws. They are widely used in computational fluid dynamics (CFD), computational acoustics, and computational magneto-hydrodynamics. By combining the advantages of the finite element (FE) and finite volume (FV), DG methods, one can achieve high order accuracy while remaining the compactness of the stencil. Meanwhile, DG methods are especially suitable for hyperbolic-type systems of equations in terms of solution accuracy,, , $3,8,36,42$ treatment of non-conforming meshes, ${ }^{16}$ and implementation of the hp-adaptivity. ${ }^{38}$ However, the DG methods have a number of their own weaknesses. In particular, how to reduce the computing costs for the DG methods, and how to efficiently solve elliptic problems or discretize diffusion terms in the parabolic equations remain the three unresolved and challenging issues in the DG methods.

In order to reduce both computational costs and storage requirements of DG methods, a new family of reconstructed DG methods, termed $\mathrm{P}_{n} \mathrm{P}_{m}$ schemes, referred as $\mathrm{rDG}\left(\mathrm{P}_{n} \mathrm{P}_{m}\right)$ in this paper, was introduced by Dumbser et al. ${ }^{10-12}$ Here, $\mathrm{P}_{n}$ indicates that a piecewise polynomial of degree of $n$ is used to represent a underlying DG solution, and $\mathrm{P}_{m}$ represents a reconstructed polynomial solution of degree of $m(m \geq n)$ that is used to compute the fluxes and source terms. Note that the $\operatorname{rDG}\left(\mathrm{P}_{n} \mathrm{P}_{m}\right)$ schemes provide a unified

[^0]formulation for both FVM and DGM, and contain both classical FVM and standard DGM as two special cases of $\mathrm{rDG}\left(\mathrm{P}_{n} \mathrm{P}_{m}\right)$ schemes. The hierarchical WENO-based rDG $\left(\mathrm{P}_{n} \mathrm{P}_{m}\right)$ schemes ${ }^{17-22}$ are designed not only to reduce the high computing costs associated with DG methods, but also to avoid spurious oscillations in the vicinity of strong discontinuities, thus effectively overcoming the first two shortcomings of the DG methods.

Indeed, DG methods are natural choices for solving hyperbolic systems, such as compressible Euler equations. However, when it comes to elliptic or parabolic equations, such as compressible Navier-Stokes equations, the DG formulation is far less certain and advantageous. Approaches made to resolve this issue could be found in the literature. ${ }^{1,2,4-6,9,13,19,34,35,39,40}$ Those methods have introduced in some way the influence of the discontinuities in order to define correct and consistent diffusive fluxes. Unfortunately, all these methods seem to require substantially more computational effort than the classical continuous finite element methods, which are naturally more suited for the discretization of elliptic problems.

As a matter of fact, over the last several years, an alternative approach for viscous discretization, which reformulates the viscous terms as a first-order hyperbolic system (FOHS), was developed by Nishikawa. ${ }^{14,23,25-27,29-31,33}$ In the FOHS formulation, by including derivative quantities as additional variables, the equations are first formulated as a first order system (FOS). Then, it is rendered to be hyperbolic, which is the distinguished feature of the FOHS method from other FOS methods, by adding pseudo time derivatives to the first-order system. It thus generates a system of pseudo-time evolution equations for the solution and the derivatives in the partial differential equation (PDE) level, not in the discretization level as in DG methods. Due to the fact that the well-established methods can be directly applied to the viscous terms in the FOHS, the formulation in the PDE level would allow a dramatic simplification in the discretization. The FOS method is especially attractive in the context of the DG methods, since it allows the use of inviscid algorithms for the viscous terms and thus greatly simplifies the discretization of the compressible Navier-Stokes equations.

A challenge in combining the DGM and the FOHS method lies in a very large number of discrete unknowns arising from both methods. For a scalar equation in two dimensions, the FOHS method introduces two derivatives as additional unknowns, and a $\mathrm{P}_{1}$ DGM introduces three degrees of freedom (solution, and derivatives) for each variables, resulting in the total of nine degrees of freedom. In 2015, the fifth author noticed that these degrees of freedom can be significantly reduced by unifying inter-related high-order moments of the derivative variables and extending the idea of Scheme-II ${ }^{29}$ to replace high-order moments of a solution polynomial by the derivative variables. He has shown that the total number of degrees of freedom can be reduced from nine to six while the order of polynomial is upgraded to quadratic for the solution variable. The resulting approximation is comparable to a $\mathrm{P}_{2}$ DGM. Therefore, if compared with a one-order higher conventional DGM, the FOHS method requires virtually no increase in the degrees of freedom. The method extends systematically to arbitrary order of accuracy: $\mathrm{P}_{k}$ hyperbolic DGM gives comparable accuracy as $\mathrm{P}_{k+1}$ DGM for the same number of degrees of freedom. Later, the method was presented formally in Ref., ${ }^{24}$ focusing on advection-dominated problems. However, the specific method described in Ref. ${ }^{24}$ is not yet an attractive approach for practical applications. First, it has one-order-lower accuracy in the diffusion term than a conventional DGM (see Table 3 in Ref. ${ }^{24}$ ), thus leading to lower order accuracy, for example, in boundary layer calculations. Second, since a direct solver is employed for solving the linear system in the Newton method, convergence acceleration by the elimination of second derivatives, which is one of the advantages of the hyperbolic method, is not achieved. Therefore, this approach is, although more efficient than a straightforward DG discretization of the FOHS, actually less efficient than conventional DG methods, not fully taking advantage of the hyperbolic method. More importantly, the method does not reduce the cost of the DGM. In this study, we explore the combination of the FOHS method and the rDG method to further reduce the cost of the DGM applied to the FOHS towards affordable high-order unstructured-grid methods for practical applications.

The objective of the effort discussed in the present work is to develop high order hyperbolic rDG methods for solving advection-diffusion equations based on the FOHS formulation, termed hyperbolic rDG methods in this paper. Different reconstruction scheme, including hybrid least-squares (LS) ${ }^{7}$ and variational reconstruction (VR), ${ }^{41}$ has been implemented in the study. By combining FOHS and rDG methods, the presented methods can provide high-order results in both primary variables and the derivatives efficiently. Same degrees of freedom are obtained as the conventional DG methods by choosing the corresponding unknown vector and basis matrix. The hyperbolic rDG method is a general framework, including finite-volume methods and the method in Ref. ${ }^{24}$ as special cases. In this study, we consider the upwind hyperbolic-diffusion flux to exploit the maximum potential of the hyperbolic method. A number of advection-diffusion problems are
presented, including boundary-layer type problems and pure diffusion problems which are not reported in Ref., ${ }^{24}$ indicating the hyperbolic rDG method is a cost-effective high-order scheme, and has the potential to ultimately be applied to the incompressible and compressible Navier-Stokes equations on fully irregular, adaptive, anisotropic, unstructured grids.

The outline of the rest of this paper is organized as follows. A FOHS formulation for advection-diffusion equations is described in Section II. The rDG methods for solving the hyperbolic diffusion equations are presented in Section III. Extensive numerical experiments are reported in Section IV. Concluding remarks and a plan of future work are given in Section V.

## II. FOHS formulation for Advection-Diffusion Equations

Consider the following model linear advection-diffusion equation in 2D.

$$
\begin{equation*}
\frac{\partial \varphi}{\partial \tau}+a \frac{\partial \varphi}{\partial x}+b \frac{\partial \varphi}{\partial y}=\nu\left(\frac{\partial^{2} \varphi}{\partial x^{2}}+\frac{\partial^{2} \varphi}{\partial y^{2}}\right)+f(x, y) \tag{1}
\end{equation*}
$$

where $\varphi$ denotes a scalar function that can be referred to as velocity potential, $(a, b)$ is a constant advection vector, $\nu$ is a positive diffusion coefficient, and $f(x, y)$ is the source term. In order to reformulate this equation into a first-order hyperbolic advection diffusion system, derivatives of the unknown variable $\varphi$ would be needed as additional variables. Therefore the velocity vector $\mathbf{u}$ is defined as

$$
\mathbf{u}=\nabla \varphi=\left[\begin{array}{ll}
\mathrm{v}_{x}, & \mathrm{v}_{y} \tag{2}
\end{array}\right]^{T}
$$

where $u$ and $v$ are the components of the velocity vector.
By adding pseudo time derivatives with respect to all variables, the following first-order hyperbolic system for this advection-diffusion equation could be formulated.

$$
\left\{\begin{align*}
\frac{\partial \varphi}{\partial \tau}+a \frac{\partial \varphi}{\partial x}+b \frac{\partial \varphi}{\partial y} & =\nu\left(\frac{\partial \mathrm{v}_{x}}{\partial x}+\frac{\partial \mathrm{v}_{y}}{\partial y}\right)+f(x, y)  \tag{3}\\
\frac{\partial \mathrm{v}_{x}}{\partial \tau} & =\frac{1}{T_{r}}\left(\frac{\partial \varphi}{\partial x}-\mathrm{v}_{x}\right) \\
\frac{\partial \mathrm{v}_{y}}{\partial \tau} & =\frac{1}{T_{r}}\left(\frac{\partial \varphi}{\partial y}-\mathrm{v}_{y}\right)
\end{align*}\right.
$$

where $\tau$ is understood as the pseudo time. Clearly, the velocity would relax to the solution derivatives in the steady state. Here $T_{r}$ is a free parameter, named as relaxation time. Note that the system is equivalent to the original advection-diffusion equation in the steady state for any nonzero $T_{r}$, but $T_{r}$ needs to be positive for the system to be hyperbolic. The system may be solved by marching in the pseudo time to yield a steady solution to the original advection-diffusion equation.

The FOHS can be written in the vector form as

$$
\begin{equation*}
\frac{\partial \mathbf{U}}{\partial \tau}+\frac{\partial \mathbf{F}_{x}}{\partial x}+\frac{\partial \mathbf{F}_{y}}{\partial y}=\mathbf{S} \tag{4}
\end{equation*}
$$

where

$$
\mathbf{U}=\left(\begin{array}{c}
\varphi  \tag{5}\\
\mathrm{v}_{x} \\
\mathrm{v}_{y}
\end{array}\right), \quad \mathbf{F}_{x}=\left(\begin{array}{c}
a \varphi-\nu \mathrm{v}_{x} \\
-\varphi / T_{r} \\
0
\end{array}\right), \quad \quad \mathbf{F}_{y}=\left(\begin{array}{c}
b \varphi-\nu \mathrm{v}_{y} \\
0 \\
-\varphi / T_{r}
\end{array}\right), \quad \mathbf{S}=\left(\begin{array}{c}
f(x, y) \\
-\mathrm{v}_{x} / T_{r} \\
-\mathrm{v}_{y} / T_{r}
\end{array}\right)
$$

In this paper, we consider the advection term and the diffusive term separately.

$$
\mathbf{F}_{x}=\mathbf{F}_{x}^{a}+\mathbf{F}_{x}^{d}=\left(\begin{array}{c}
a \varphi  \tag{6}\\
0 \\
0
\end{array}\right)+\left(\begin{array}{c}
-\nu \mathrm{v}_{x} \\
-\varphi / T_{r} \\
0
\end{array}\right), \quad \mathbf{F}_{y}=\mathbf{F}_{y}^{a}+\mathbf{F}_{y}^{d}=\left(\begin{array}{c}
b \varphi \\
0 \\
0
\end{array}\right)+\left(\begin{array}{c}
-\nu \mathrm{v}_{y} \\
0 \\
-\varphi / T_{r}
\end{array}\right)
$$

Consider the Jacobian of the flux projected along $\mathbf{n}=\left(n_{x}, \quad n_{y}\right)$,

$$
\begin{equation*}
\mathbf{A}_{n}=\frac{\partial \mathbf{F}_{x}}{\partial \mathbf{U}} n_{x}+\frac{\partial \mathbf{F}_{y}}{\partial \mathbf{U}} n_{y}=\mathbf{A}_{n}^{a}+\mathbf{A}_{n}^{d} \tag{7}
\end{equation*}
$$

where $\mathbf{A}_{n}^{a}$ and $\mathbf{A}_{n}^{d}$ are the advective and diffusiove Jacobians, respectively.

$$
\mathbf{A}_{n}^{a}=\frac{\partial \mathbf{F}_{x}^{a}}{\partial \mathbf{U}} n_{x}+\frac{\partial \mathbf{F}_{y}^{a}}{\partial \mathbf{U}} n_{y}=\left(\begin{array}{ccc}
a_{n} & 0 & 0  \tag{8}\\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right), \mathbf{A}_{n}^{d}=\frac{\partial \mathbf{F}_{x}^{d}}{\partial \mathbf{U}} n_{x}+\frac{\partial \mathbf{F}_{y}^{d}}{\partial \mathbf{U}} n_{y}=\left(\begin{array}{ccc}
0 & -\nu n_{x} & -\nu n_{y} \\
-n_{x} / T_{r} & 0 & 0 \\
-n_{y} / T_{r} & 0 & 0
\end{array}\right)
$$

and

$$
\begin{equation*}
a_{n}=a n_{x}+b n_{y} \tag{9}
\end{equation*}
$$

The only non-zero eigenvalue of advective Jacobian is $a_{n}$, while the diffusive Jacobian has the following eigenvalues

$$
\begin{equation*}
\lambda_{1}=\sqrt{\frac{\nu}{T_{r}}}, \quad \lambda_{2}=-\sqrt{\frac{\nu}{T_{r}}}, \quad \lambda_{3}=0 \tag{10}
\end{equation*}
$$

The first two nonzero eigenvalues indicate that the system describes a wave propagating isotropically if we only consider the diffusive part. The third eigenvalue corresponds to the inconsistency damping mode. The relaxation time $T_{r}$ does not affect the steady solution, and thus can be defined solely for the purpose of accelerating the convergence to the steady state. For simplicity, $T_{r}$ is defined as

$$
\begin{equation*}
T_{r}=\frac{L_{r}^{2}}{\nu}, \quad L_{r}=\frac{1}{\max (\operatorname{Re}, 2 \pi)}, \quad \operatorname{Re}=\frac{\sqrt{a^{2}+b^{2}}}{\nu} \tag{11}
\end{equation*}
$$

Note that we include Reynolds number information in the relaxation length scale, so that the developed method could deliver the designed order of accuracy with fast convergence when it comes to narrow boundary layer type problem. ${ }^{32}$

## III. Reconstruction Discontinuous Galerkin Methods

The above generated FOHS of equations, i.e. Eq. (4) can be discretized using a discontinuous Galerkin finite element formulation. We assume that the domain $\Omega$ is subdivided into a collection of non-overlapping arbitrary elements $\Omega_{e}$, and then introduce the following broken Sobolev space $V_{h}^{n}$

$$
\begin{equation*}
V_{h}^{n}=\left\{v_{h} \in\left[L^{2}(\Omega)\right]^{k}:\left.v_{h}\right|_{\Omega_{e}} \in\left[V_{n}^{k}\right] \forall \Omega_{e} \in \Omega\right\} \tag{12}
\end{equation*}
$$

which consists of discontinuous vector polynomial functions of degree $n$, and where $k$ is the dimension of the unknown vector and $V_{n}$ is the space of all polynomials of degree $\leq n$. To formulate the discontinuous Galerkin method, we introduce the following weak formulation, which is obtained by multiplying Eq. (4) by a test function $\mathbf{W}_{h}$, integrating over an element $\Omega_{e}$, and then performing an integration by parts: find $\mathbf{U}_{h} \in V_{h}^{p}$ such as

$$
\begin{equation*}
\frac{d}{d \tau} \int_{\Omega_{e}} \mathbf{W}_{h} \mathbf{U}_{h} d \Omega+\int_{\Gamma_{e}} \mathbf{W}_{h} \mathbf{F}_{k} \mathbf{n}_{k} d \Gamma-\int_{\Omega_{e}} \frac{\partial \mathbf{W}_{h}}{\partial x_{k}} \mathbf{F}_{k} d \Omega=\int_{\Omega_{e}} \mathbf{W}_{h} \mathbf{S} d \Omega, \quad \forall \mathbf{W}_{h} \in V_{h}^{n} \tag{13}
\end{equation*}
$$

where $\mathbf{U}_{h}$ and $\mathbf{W}_{h}$ are represented by piecewise polynomial functions of degrees $p$, which are discontinuous between the cell interfaces, and $\mathbf{n}_{k}$ the unit outward normal vector to the $\Gamma_{e}$ : the boundary of $\Omega_{e}$. With the method of lines, the standard DG solution $\mathbf{U}_{h}$ within the element $E$ can be expressed as

$$
\begin{equation*}
\mathbf{U}_{h}(x, y, \tau)=\mathbf{C}(x, y) \mathbf{V}(\tau) \tag{14}
\end{equation*}
$$

where $\mathbf{C}$ is a basis matrix, and $\mathbf{V}$ is a vector of unknown polynomial coefficients. We will discuss a little bit more about $\mathbf{C}$ and $\mathbf{V}$ later in this section.

If we set the test function $\mathbf{W}_{h}$ as the transpose of the basis matrix $\mathbf{C}$, then the following equivalent system would be arrived.

$$
\begin{equation*}
\frac{d}{d \tau} \int_{\Omega_{e}} \mathbf{C}^{T} \mathbf{C V} d \Omega+\int_{\Gamma_{e}} \mathbf{C}^{T} \mathbf{F}_{k} \mathbf{n}_{k} d \Gamma-\int_{\Omega_{e}} \frac{\partial \mathbf{C}^{T}}{\partial x_{k}} \mathbf{F}_{k} d \Omega=\int_{\Omega_{e}} \mathbf{C}^{T} \mathbf{S} d \Omega \tag{15}
\end{equation*}
$$

Since the numerical solution $\mathbf{U}_{h}$ is discontinuous between element interfaces, the interface fluxes are not uniquely defined. This scheme is called discontinuous Galerkin method of degree $n$, or in short notation
$\mathrm{DG}\left(\mathrm{P}_{n}\right)$ method. By simply increasing the degree $n$ of the polynomials, the DG methods of corresponding higher order are obtained.

Compared with reconstructed FV methods, the DG methods would require more degrees of freedom, additional domain integration, and more Gauss quadrature points for the boundary integration, which leads to more computational costs and storage requirements. Inspired by the reconstructed DG methods from Dumbser et al. in the frame of $\mathrm{P}_{n} \mathrm{P}_{m}$ scheme, ${ }^{10-12}$ termed $\mathrm{rDG}\left(\mathrm{P}_{n} \mathrm{P}_{m}\right)$ in this paper, least-squares based and variational reconstruction based rDG methods are designed to achieve high order of accuracy while reducing the computational cost. In fact, a unified formulation would be provided by rDG method for both FV and DG methods. The standard FV and DG method would be nothing but special cases in rDG framework, and thus allow for a direct efficiency comparison. For $\mathrm{rDG}\left(\mathrm{P}_{n} \mathrm{P}_{m}\right)$ method with $m>n$, a higher-order reconstructed numerical solution can be obtained:

$$
\begin{equation*}
\mathbf{U}_{h}^{R}(x, y, \tau)=\mathbf{C}^{R}(x, y) \mathbf{V}^{R}(\tau) \tag{16}
\end{equation*}
$$

where higher-order derivatives (higher than $n$-th and up to $m$-th) are reconstructed from the underlying $\mathrm{P}_{n}$ solution. There are three approaches to the reconstruction. One is a least-squares reconstruction method, and another is a variational reconstruction method. The last option, which is unique in the FOHS formulation considered here, is to directly use the gradient variables and their moments to evaluate these derivatives. Or equivalently, this approach can be thought of as defining the solution as $\mathrm{P}_{m}$, and use higher-order moments to represent the gradient variables in the FOHS formulation. In the former two approaches, the method is expressed by $\operatorname{rDG}\left(\mathrm{P}_{n} \mathrm{P}_{m}\right)$, and the latter approach by $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{m}\right)$ since high-order derivatives are already available and no explicit reconstruction is required. This higher order numerical solution $\mathbf{U}_{h}^{R}$ would be used for flux and source term computation.

By moving the second and third terms to the right-hand-side (r.h.s.) in Eq. (15), we will arrive at a system of ordinary differential equations (ODEs) in time, which can be written in semi-discrete form as

$$
\begin{equation*}
\mathbf{M} \frac{d \mathbf{V}}{d \tau}=\mathbf{R}\left(\mathbf{U}_{h}^{R}\right) \tag{17}
\end{equation*}
$$

where $\mathbf{M}$ is the mass matrix,

$$
\begin{equation*}
\mathbf{M}=\int_{\Omega_{e}} \mathbf{C}^{T} \mathbf{C} d \Omega \tag{18}
\end{equation*}
$$

and $\mathbf{R}$ is the residual vector, defined as

$$
\begin{equation*}
\mathbf{R}=\int_{\Omega_{e}} \frac{\partial \mathbf{C}^{T}}{\partial x_{k}} \mathbf{F}_{k}\left(\mathbf{U}_{h}^{R}\right)+\mathbf{C}^{T} \mathbf{S}\left(\mathbf{U}_{h}^{R}\right) d \Omega-\int_{\Gamma_{e}} \mathbf{C}^{T} \mathbf{F}_{k}\left(\mathbf{U}_{h}^{R}\right) \mathbf{n}_{k} d \Gamma \tag{19}
\end{equation*}
$$

Based on different rDG methods, some effective discretization hyperbolic rDG methods will be presented to deal with the derived FOHS. The format $\mathbf{A}+\mathbf{B}$ is used to indicate the discretization method for the system, where $\mathbf{A}$ refers to the discretization method for $\varphi$ and $\mathbf{B}$ refers to the discretization method for its derivatives. Different choices and combinations for $\mathbf{A}$ and $\mathbf{B}$ are compared in the authors' previous work. ${ }^{15}$ To minimize the memory and storage cost of the developed methods, one can apply $\mathrm{DG}\left(\mathrm{P}_{n}\right)$ or $\mathrm{rDG}\left(\mathrm{P}_{n} \mathrm{P}_{m}\right)$ methods only on the derivative variables. With the handily information of the derivatives, a higher order of polynomial for $\varphi$ can be constructed with only one degree of freedom. Therefore, in this paper, we would focus on $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{n+1}\right)+\mathrm{DG}\left(\mathrm{P}_{n}\right)$ and $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{m+1}\right)+\mathrm{rDG}\left(\mathrm{P}_{n} \mathrm{P}_{m}\right)$ methods.

In the implementation of the DG methods in this paper, modal based DG methods are adopted. The numerical polynomial solutions are represented using a Taylor series expansion at the cell center and normalized to improve the conditioning of the system matrix. For instance, according to the Taylor expansion in 2 D , one would have

$$
\begin{align*}
\varphi_{h} & =\bar{\varphi}+\varphi_{x}^{c} \Delta x B_{2}+\varphi_{y}^{c} \Delta y B_{3}+\varphi_{x x}^{c} \Delta x^{2} B_{4}+\varphi_{y y}^{c} \Delta y^{2} B_{5}+\varphi_{x y}^{c} \Delta x \Delta y B_{6} \\
& +\varphi_{x x x}^{c} \Delta x^{3} B_{7}+\varphi_{y y y}^{c} \Delta y^{3} B_{8}+\varphi_{x x y}^{c} \Delta x^{2} \Delta y B_{9}+\varphi_{x y y}^{c} \Delta x \Delta y^{2} B_{10}+\cdots \tag{20}
\end{align*}
$$

where the $\bar{\varphi}$ represents the averaged quantity of $\varphi$, and the superscript $c$ stands for the central values. The
basis functions are given as follows

$$
\begin{gather*}
B_{1}=1, B_{2}=\frac{x-x_{c}}{\Delta x}, B_{3}=\frac{y-y_{c}}{\Delta y}, \\
B_{4}=\frac{1}{2}\left(B_{2}^{2}-\frac{1}{\Omega_{e}} \int_{\Omega_{e}} B_{2}^{2} d \Omega\right), B_{5}=\frac{1}{2}\left(B_{3}^{2}-\frac{1}{\Omega_{e}} \int_{\Omega_{e}} B_{3}^{2} d \Omega\right), B_{6}=B_{2} B_{3}-\frac{1}{\Omega_{e}} \int_{\Omega_{e}} B_{2} B_{3} d \Omega, \\
B_{7}=\frac{1}{6}\left(B_{2}^{3}-\frac{1}{\Omega_{e}} \int_{\Omega_{e}} B_{2}^{3} d \Omega\right), B_{8}=\frac{1}{6}\left(B_{3}^{3}-\frac{1}{\Omega_{e}} \int_{\Omega_{e}} B_{3}^{3} d \Omega\right),  \tag{21}\\
B_{9}=\frac{1}{2}\left(B_{2}^{2} B_{3}-\frac{1}{\Omega_{e}} \int_{\Omega_{e}} B_{2}^{2} B_{3} d \Omega\right), B_{10}=\frac{1}{2}\left(B_{2} B_{3}^{2}-\frac{1}{\Omega_{e}} \int_{\Omega_{e}} B_{2} B_{3}^{2} d \Omega\right) .
\end{gather*}
$$

Here, we have

$$
\begin{equation*}
\Delta x=0.5\left(x_{\max }-x_{\min }\right), \quad \Delta y=0.5\left(y_{\max }-y_{\min }\right) \tag{22}
\end{equation*}
$$

where $x_{\max }, x_{\min }, y_{\max }$, and $y_{\min }$ are used to represent the maximum and minimum coordinates values of the vertexes of the cell.

As we mentioned in the previous work, ${ }^{15}$ one can choose the unknown vector $\mathbf{V}$ to make the resultant scheme have same degrees of freedom as conventional DG methods. As a matter of fact, if a PetrovGalerkin formulation with a simplified basis function matrix is implemented, one would end up with the same hyperbolic rDG methods we used for diffusion equation. ${ }^{15}$ On the other hand, if a consistent Galerkin formulation is used, one can make all variables coupled and thus to have better stability properties for most advection-diffusion problem. Here, some examples of the unknown vector $\mathbf{V}$ and the basis matrix $\mathbf{C}$ under Galerkin formulation are shown for better illustration.

- $\operatorname{DG}\left(\mathrm{P}_{0} \mathrm{P}_{1}\right)+\mathrm{DG}\left(\mathrm{P}_{0}\right)$

$$
\begin{gather*}
\mathbf{V}=\left(\begin{array}{c}
\bar{\varphi} \\
\overline{\varphi_{x}} \Delta x \\
\overline{\varphi_{y}} \Delta y
\end{array}\right)  \tag{23}\\
\mathbf{C}=\left(\begin{array}{ccc}
B_{1} & B_{2} & B_{3} \\
0 & B_{1} \Delta x^{-1} & 0 \\
0 & 0 & B_{1} \Delta y^{-1}
\end{array}\right) . \tag{24}
\end{gather*}
$$

As we can see here, the basis matrix $\mathbf{V}$ has included the connection between $\varphi$ and its derivatives, leading to a coupled system. Note that the derivatives of $\varphi$ are determined as solutions to the FOHS, whereas conventional $P_{1}$ DG methods determine them as solutions to discrete equations derived by the weak formulation.

- $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{2}\right)+\mathrm{rDG}\left(\mathrm{P}_{0} \mathrm{P}_{1}\right)$

$$
\begin{gather*}
\mathbf{V}=\left(\begin{array}{c}
\bar{\varphi} \\
\overline{\varphi_{x}} \Delta x \\
\overline{\varphi_{y}} \Delta y
\end{array}\right)  \tag{25}\\
\mathbf{C}=\left(\begin{array}{ccc}
B_{1} & B_{2} & B_{3} \\
0 & B_{1} \Delta x^{-1} & 0 \\
0 & 0 & B_{1} \Delta y^{-1}
\end{array}\right)  \tag{26}\\
\mathbf{V}^{R}=\left[\bar{\varphi}, \overline{\varphi_{x}} \Delta x, \overline{\left.\varphi_{x} \Delta y, \varphi_{x x}^{c, R} \Delta x^{2}, \varphi_{y y}^{c, R} \Delta y^{2}, \varphi_{x y}^{c, R} \Delta x \Delta y\right]^{T},}\right.  \tag{27}\\
\mathbf{C}^{R}=\left(\begin{array}{cccccc}
B_{1} & B_{2} & B_{3} & B_{4} & B_{5} & B_{6} \\
0 & B_{1} \Delta x^{-1} & 0 & B_{2} \Delta x^{-1} & 0 & B_{3} \Delta x^{-1} \\
0 & 0 & B_{1} \Delta y^{-1} & 0 & B_{3} \Delta y^{-1} & B_{2} \Delta y^{-1}
\end{array}\right) . \tag{28}
\end{gather*}
$$

This hyperbolic rDG method would have same $\mathbf{C}$ and $\mathbf{V}$ as $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{1}\right)+\mathrm{DG}\left(\mathrm{P}_{0}\right)$, since they have the same orders for the underlying DG solution. As a result, the mass matrices would be the same as well. However, a higher order polynomial $\mathbf{U}_{h}^{R}$ would be used for computing the flux and source term, and thus to yield a more accurate solution. The higher order terms, i.e., $\varphi_{x x}^{c, R} \Delta x^{2}, \varphi_{y y}^{c, R} \Delta y^{2}, \varphi_{x y}^{c, R} \Delta x \Delta y$ are computed using reconstruction schemes. For example, in this study, a hybrid least-squares scheme (LS) ${ }^{7}$ and a variational reconstruction scheme (VR) ${ }^{41}$ have been implemented to obtain higher moments.

- $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{2}\right)+\mathrm{DG}\left(\mathrm{P}_{1}\right)$

$$
\begin{gather*}
\mathbf{V}=\left[\bar{\varphi}, \overline{\varphi_{x}} \Delta x, \overline{\varphi_{x}} \Delta y, \varphi_{x x}^{c} \Delta x^{2}, \varphi_{y y}^{c} \Delta y^{2}, \varphi_{x y}^{c} \Delta x \Delta y\right]^{T},  \tag{29}\\
\mathbf{C}=\left(\begin{array}{cccccc}
B_{1} & B_{2} & B_{3} & B_{4} & B_{5} & B_{6} \\
0 & B_{1} \Delta x^{-1} & 0 & B_{2} \Delta x^{-1} & 0 & B_{3} \Delta x^{-1} \\
0 & 0 & B_{1} \Delta y^{-1} & 0 & B_{3} \Delta y^{-1} & B_{2} \Delta y^{-1}
\end{array}\right) . \tag{30}
\end{gather*}
$$

Compared with authors' previous work, ${ }^{15}$ the degrees of freedom for $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{2}\right)+\mathrm{DG}\left(\mathrm{P}_{1}\right)$ has reduced from 7 to 6 by replacing the redundant cross term with a unified unknown. Hence, this method has the same number of degrees of freedom as P2 conventional DG methods.

- $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{3}\right)+\mathrm{DG}\left(\mathrm{P}_{2}\right)$

$$
\begin{equation*}
\mathbf{V}=\left[\bar{\varphi}, \overline{\varphi_{x}} \Delta x, \overline{\varphi_{x}} \Delta y, \varphi_{x x}^{c} \Delta x^{2}, \varphi_{y y}^{c} \Delta y^{2}, \varphi_{x y}^{c} \Delta x \Delta y, \varphi_{x x x}^{c} \Delta x^{3}, \varphi_{y y y}^{c} \Delta y^{3}, \varphi_{x x y}^{c} \Delta x^{2} \Delta y, \varphi_{x y y}^{c} \Delta x \Delta y^{2}\right]^{T}, \tag{31}
\end{equation*}
$$

$$
\mathbf{C}^{T}=\left(\begin{array}{ccc}
B_{1} & 0 & 0  \tag{32}\\
B_{2} & B_{1} \Delta x^{-1} & 0 \\
B_{3} & 0 & B_{1} \Delta y^{-1} \\
B_{4} & B_{2} \Delta x^{-1} & 0 \\
B_{5} & 0 & B_{3} \Delta y^{-1} \\
B_{6} & B_{3} \Delta x^{-1} & B_{2} \Delta y^{-1} \\
B_{7}-B_{2} B_{4}^{c} & B_{4} \Delta x^{-1} & 0 \\
B_{8}-B_{3} B_{5}^{c} & 0 & B_{5} \Delta y^{-1} \\
B_{9}-B_{2} B_{6}^{c}-B_{3} B_{4}^{c} & B_{6} \Delta x^{-1} & B_{4} \Delta y^{-1} \\
B_{1} 0-B_{2} B_{5}^{c}-B_{3} B_{6}^{c} & B_{5} \Delta x^{-1} & B_{6} \Delta y^{-1}
\end{array}\right)^{T} .
$$

The complexity in the basis matrix is due to the fact that the average values and cell centers values are not equal. This can be derived using Taylor expansion. One can find a similar procedure in Ref. ${ }^{24}$

Classically, the conventional DG would need two numerical flux schemes to solve the advection-diffusion equation. While DG methods naturally developed for hyperbolic equations, the diffusive flux are not that straightforward or efficient. However, with FOHS, rDG method could use well-established methods for hyperbolic systems. In this paper, the simplest upwind method is applied for the numerical flux across the interface.

$$
\begin{equation*}
\mathbf{F}_{i j}=\frac{1}{2}\left(\mathbf{F}_{L}+\mathbf{F}_{R}\right) \cdot \mathbf{n}_{i j}-\frac{1}{2}\left(\left|\mathbf{A}_{n}^{a}\right|+\left|\mathbf{A}_{n}^{d}\right|\right)\left(\mathbf{U}_{R}-\mathbf{U}_{L}\right) . \tag{33}
\end{equation*}
$$

where $\mathbf{n}_{i j}$ is the unit directed area vector, and $\left|\mathbf{A}^{a}\right|$ and $\left|\mathbf{A}^{d}\right|$ would be

$$
\left|\mathbf{A}_{n}^{a}\right|=\left(\begin{array}{ccc}
\left|a_{n}\right| & 0 & 0  \tag{34}\\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right), \quad\left|\mathbf{A}_{n}^{d}\right|=\frac{\nu}{L_{r}}\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & n_{x}^{2} & n_{x} n_{y} \\
0 & n_{x} n_{y} & n_{y}^{2}
\end{array}\right) .
$$

Note that, the absolute Jacobian is constructed independently for both advection and diffusion terms. Here we are not assuming $\left|\mathbf{A}_{n}\right|=\left|\mathbf{A}_{n}^{a}\right|+\left|\mathbf{A}_{n}^{d}\right|$. What we did here is an approximation, which would allow us to avoid the analysis for the eigen-structure for the whole system. As a matter of fact, for this simple advection-diffusion equation, the eigen-structure for the whole system is still analyzable with some extra effort. However, when it comes to complex conservation laws, such as the Navier-Stokes equations, one can only rely on the approximation approach at present. As a matter of fact, this method has been successfully demonstrated for the Navier-Stokes system. ${ }^{28}$

Boundary conditions are enforced weakly through the numerical flux in a similar manner as in the previous work. ${ }^{15}$ For all test problems, the Dirichlet condition is considered, and therefore only the solution $\varphi$ is given on boundaries. At a boundary face, $\mathbf{n}_{i j}$ is taken to be outward, and thus $\mathbf{U}_{R}$ is considered as a boundary state. The boundary condition is incorporated into the boundary state as

$$
\begin{equation*}
\mathbf{U}_{R}=\left(\varphi_{b}, \mathrm{v}_{n} n_{x}+\partial_{s} \varphi_{b} t_{x}, \mathrm{v}_{n} n_{y}+\partial_{s} \varphi_{b} t_{y}\right), \tag{35}
\end{equation*}
$$

where $\varphi_{b}$ is the value given as a boundary condition, and $\partial_{s} \varphi_{b}$ is the tangential derivative that can be obtained from the given boundary condition, $\mathrm{v}_{n}$ is the face-normal projection of $\left(\mathrm{v}_{x}, \mathrm{v}_{y}\right)$ evaluated at the left (interior) state $\mathbf{U}_{L}, \mathbf{n}_{i j}=\left(n_{x}, n_{y}\right)$, and $\left(t_{x}, t_{y}\right)$ denotes a unit tangent vector of the boundary face. Note that $s$ is taken to be positive in the counterclockwise direction along a boundary, and the tangent vector is also taken in the same direction. In the case of a unit square domain, the boundary state becomes

$$
\begin{equation*}
\mathbf{U}_{R}=\left(\varphi_{b}, \mathrm{v}_{x}, \partial_{y} \varphi_{b}\right) \tag{36}
\end{equation*}
$$

at the left and right boundaries, and

$$
\begin{equation*}
\mathbf{U}_{R}=\left(\varphi_{b}, \partial_{x} \varphi_{b}, \mathbf{v}_{y}\right) \tag{37}
\end{equation*}
$$

at the top and bottom boundaries.
Note that the normal component $\mathrm{v}_{n}$ may be specified in place of $\varphi_{b}$ in the case of Neumann problems. As discussed in, ${ }^{26}$ the hyperbolic diffusion system has one wave going out of the domain, and therefore one quantity should be left unspecified, which corresponds to the normal derivative $\mathrm{v}_{n}$ in the Dirichlet case (or $\varphi$ in the Neumann case). Or it may be argued that since the hyperbolic diffusion system is equivalent to the original diffusion equation in the pseudo steady state, the boundary condition should also be the same as the original problem. The tangential derivative can be specified since $\varphi$ is known in the Dirichlet case, but it is not necessary; the results are very similar with and without specifying $\partial_{s} \varphi_{b}$.

A steady solution can be obtained for marching in the pseudo time. In this paper, BDF1 scheme has been employed in the work, with the local time step defined as

$$
\begin{equation*}
\Delta \tau=\mathrm{CFL} \frac{2 \Omega_{e}}{\sum_{j}\left(\nu / L_{r}+\Omega_{j} / T_{r}\right)} \tag{38}
\end{equation*}
$$

The semi-discrete system of ordinary differential equations, i.e., Eq. (17) can be linearized as

$$
\begin{equation*}
\left(\frac{\mathbf{M}}{\Delta \tau}-\frac{\partial \mathbf{R}}{\partial \mathbf{V}}\right) \Delta \mathbf{V}_{\mathbf{i}}=\mathbf{R}\left(\mathbf{U}_{\mathbf{h}}^{\mathbf{R}}\right) \tag{39}
\end{equation*}
$$

This system represents a system of linear simultaneous algebraic equations and needs to be solved at each time step. The most widely used methods to solve this linear system are iterative solution methods and approximate factorization methods. In this study,GMRES+LU-SGS and GCR + SGS (k) have been developed to solve the linear system, where LU-SGS/SGS(k) serve as the preconditioner, where $k$ is the number of relaxations.

## IV. Numerical Examples

## A. 1D Boundary layer problem

In the first test case, we consider the following 1D problem

$$
\begin{equation*}
\frac{\partial \varphi}{\partial t}+a \frac{\partial \varphi}{\partial x}=\nu \frac{\partial^{2} \varphi}{\partial x^{2}}+f(x), \quad 0 \leq x \leq 1 \tag{40}
\end{equation*}
$$

with

$$
\begin{equation*}
\varphi(0)=\varphi(1)=1 \tag{41}
\end{equation*}
$$

and the source term $f(x)$ is given as

$$
\begin{equation*}
f(x)=\frac{\pi}{\operatorname{Re}}(a \cos (\pi x)+\pi \nu \sin (\pi x)), \quad \operatorname{Re}=\frac{a}{\nu} \tag{42}
\end{equation*}
$$

The exact steady solution to the problem is

$$
\begin{equation*}
\varphi(x)=\frac{\exp (-\operatorname{Re}-\exp (x \operatorname{Re}-\operatorname{Re}))}{\exp (-\operatorname{Re})-1}+\frac{1}{\operatorname{Re}} \sin (\pi x) \tag{43}
\end{equation*}
$$

The exact solution can be regard as a function of Reynolds number. In the diffusion limit, it would be a smooth sine curve, while developing a very narrow boundary layer near $x=1$ if advection limit is approached.

Based on the FOHS formulation, we would apply the developed hyperbolic rDG methods to solve the following equivalent system.

$$
\begin{equation*}
\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{F}}{\partial x}=\mathbf{S} \tag{44}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathbf{U}=\binom{\varphi}{\mathrm{v}_{x}}, \quad \mathbf{F}=\binom{a \varphi-\nu \mathrm{v}_{x}}{-\varphi / T_{r}}, \quad \mathbf{S}=\binom{f(x)}{-\mathrm{v}_{x} / T_{r}} \tag{45}
\end{equation*}
$$

In order to capture the boundary layer, one would need enough resolution in the layer. Thus, numerical experiments are carried out with non-uniform grids generated from a uniform grid by the following mapping

$$
\begin{equation*}
x_{i}=\frac{1-\exp \left(-\alpha \xi_{i}\right)}{1-\alpha}, \quad \xi_{i}=\frac{i-1}{\text { Nelem }} . \tag{46}
\end{equation*}
$$

For high Reynolds number case, or in other words, in the advection limit, one would need to increase $\alpha$ to ensure the convergence. In this paper, we set $a=1$, and all numerical results were obtained for a wide range of the Reynolds numbers, $\operatorname{Re}=10^{k}$, where $k=-8,0,8$ with varying $\nu$. And the corresponding $\alpha$ would be set as $4.5,4.5,22.5$. And the number of the elements is set to be $32,64,128$, and 256 for all Reynolds number.

Several hyperbolic rDG methods are applied here. For cases with smaller Reynolds number ( $\mathrm{Re}=1$ and $\operatorname{Re}=10^{-8}$ ), all presented method could obtain deigned order of accuracy. However, for high Reynolds number case, very strong boundary layer would arise near $x=1$. Many hyperbolic methods shown would become unstable. This result is consistent with Nishikawa's finding. ${ }^{32}$ The presented methods are in the same family of the Scheme II in the literature, ${ }^{32}$ which directly use the derivatives information to construct higher order polynomials for the primary variables. A negative diffusion coefficient would be introduced by this type of method, which leads to a lost of the upwind dissipation matrix for large mesh-Reynoldsnumbers, resulting accuracy and convergence problems. In the literature, ${ }^{32}$ the property could be improved by modifying the relaxation length scale and the treatment to the boundary condition. However, there is still remaining stability and convergence issue if one explicitly uses the higher order moments of gradients to construct high order primary variables. Instead, if one chooses independent higher moments for primary variables and the gradients, i.e., $\mathrm{DG}\left(\mathrm{P}_{n}\right)+\mathrm{DG}\left(\mathrm{P}_{n}\right)$ or $\mathrm{rDG}\left(\mathrm{P}_{n} \mathrm{P}_{m}\right)+\mathrm{rDG}\left(\mathrm{P}_{n} \mathrm{P}_{m}\right),{ }^{15}$ one can have accurate and robust result for this narrow boundary type problem. Another alternative is listed as Scheme IQ in the Ref., ${ }^{32}$ which provides robust and accurate hyperbolic schemes for boundary layer type problem. Also, using a unified eigen-structure to construct the dissipation matrix may also improve the developed hyperbolic rDG method. ${ }^{32}$ However, these approaches are beyond this paper's scope.

Table 1: Order of accuracy with different Re.

|  | Advection$\nu=10^{-8}, \operatorname{Re}=10^{8}$ |  | Advection-Diffusion$\nu=1, \operatorname{Re}=1$ |  | Diffusion$\nu=10^{8}, \operatorname{Re}=10^{-8}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\varphi$ | $\mathrm{v}_{x}$ | $\varphi$ | $\mathrm{v}_{x}$ | $\varphi$ | $\mathrm{v}_{x}$ |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{1}\right)+\mathrm{DG}\left(\mathrm{P}_{0}\right)$ | 0.91 | 0.98 | 1.00 | 1.00 | 1.01 | 1.00 |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{2}\right)+\mathrm{DG}\left(\mathrm{P}_{1}\right)$ | - | - | 2.00 | 2.00 | 2.00 | 2.00 |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{3}\right)+\mathrm{DG}\left(\mathrm{P}_{2}\right)$ | - | - | 4.01 | 3.00 | 3.96 | 3.00 |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{2}\right)+\mathrm{rDG}$ _LS $\left(\mathrm{P}_{0} \mathrm{P}_{1}\right)$ | 2.00 | 2.14 | 1.97 | 2.05 | 1.97 | 2.04 |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{3}\right)+\mathrm{rDG}$ _LS $\left(\mathrm{P}_{1} \mathrm{P}_{2}\right)$ | - | - | 3.12 | 2.68 | 3.11 | 2.69 |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{2}\right)+$ rDG_VR $\left(\mathrm{P}_{0} \mathrm{P}_{1}\right)$ | 2.00 | 1.90 | 2.00 | 2.05 | 2.00 | 2.05 |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{3}\right)+$ rDG_VR( $\left.\mathrm{P}_{0} \mathrm{P}_{2}\right)$ | 3.96 | 3.02 | 3.99 | 3.69 | 4.06 | 3.66 |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{3}\right)+\mathrm{rDG}-\mathrm{VR}\left(\mathrm{P}_{1} \mathrm{P}_{2}\right)$ | - | - | 3.89 | 3.06 | 3.80 | 3.04 |



Figure 1: Grid refinement study for $\operatorname{Re}=10^{-8}$.


Figure 2: Grid refinement study for $\operatorname{Re}=1$.


Figure 3: Grid refinement study for $\operatorname{Re}=10^{8}$.

## B. 2D steady advection diffusion problem

A steady model advection diffusion problem in a unit square is considered in this section, i.e.,

$$
\begin{equation*}
\frac{\partial \varphi}{\partial t}+a \frac{\partial \varphi}{\partial x}+b \frac{\partial \varphi}{\partial y}=\nu\left(\frac{\partial^{2} \varphi}{\partial x^{2}}+\frac{\partial^{2} \varphi}{\partial y^{2}}\right) \tag{47}
\end{equation*}
$$

with the exact solution given by

$$
\begin{equation*}
\varphi(x, y)=C \cos (A \pi \eta) \exp \left(\frac{1-\sqrt{1+4 A^{2} \pi^{2} \nu^{2}}}{2 \nu} \xi\right) \tag{48}
\end{equation*}
$$

and

$$
\begin{equation*}
\xi=a x+b y, \quad \eta=b x-a y \tag{49}
\end{equation*}
$$

In this paper, we would take $(a, b)=(2,1), A=2, C=-0.009$ with $\nu=10^{-8}, 10^{0}, 10^{8}$ to test the convergence rate for different hyperbolic rDG methods. Three sets of meshes would be used in the test, namely regular, irregular and heterogeneous grids. The sample of each type of grids are shown in Figure 4.

The grid refinement study has been carried out for the hyperbolic rDG methods under Scheme II. In each type of mesh, the advection limit case $\left(\nu=10^{-8}\right)$, advection-diffusion case $(\nu=1)$, and the diffusion limit case $\left(\nu=10^{8}\right)$ are studied, with the results shown in Table 2 to 4 and Figure 5 to 13.


Figure 4: The sample mesh of each type, i.e., $17 \times 17$ regular grid (left), $17 \times 17$ irregular grid (middle), and $23 \times 21$ heterogeneous grid (right).

Table 2: Order of accuracy on regular grids with different $\nu$.

|  | Advection$\nu=10^{-8}, \operatorname{Re}=\sqrt{5} \times 10^{8}$ |  | Advection-Diffusion$\nu=1, \operatorname{Re}=\sqrt{5}$ |  | Diffusion$\nu=10^{8}, \operatorname{Re}=\sqrt{5} \times 10^{-8}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\varphi$ | $\mathrm{v}_{x}$ | $\varphi$ | $\mathrm{v}_{x}$ | $\varphi$ | $\mathrm{v}_{x}$ |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{1}\right)+\mathrm{DG}\left(\mathrm{P}_{0}\right)$ | 2.11 | 0.99 | 1.26 | 1.00 | 1.95 | 1.00 |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{2}\right)+\mathrm{DG}\left(\mathrm{P}_{1}\right)$ | 3.02 | 2.01 | 2.03 | 1.62 | 2.05 | 1.62 |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{3}\right)+\mathrm{DG}\left(\mathrm{P}_{2}\right)$ | 3.97 | 2.97 | 3.65 | 2.99 | 3.59 | 2.97 |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{2}\right)+$ rDG_LS $\left(\mathrm{P}_{0} \mathrm{P}_{1}\right)$ | 3.18 | 2.01 | - | - | - | - |
| DG $\left(\mathrm{P}_{0} \mathrm{P}_{3}\right)+$ rDG_LS $\left(\mathrm{P}_{1} \mathrm{P}_{2}\right)$ | 4.14 | 3.22 | 3.69 | 2.82 | 3.70 | 2.82 |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{2}\right)+\mathrm{rDG}-\mathrm{VR}\left(\mathrm{P}_{0} \mathrm{P}_{1}\right)$ | 3.11 | 2.00 | 2.88 | 2.17 | 2.88 | 2.18 |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{3}\right)+\mathrm{rDG}$-VR $\left(\mathrm{P}_{0} \mathrm{P}_{2}\right)$ | 4.49 | 3.28 | 3.04 | 2.90 | 2.99 | 2.81 |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{3}\right)+\mathrm{rDG}$-VR $\left(\mathrm{P}_{1} \mathrm{P}_{2}\right)$ | 4.30 | 3.06 | 3.74 | 3.12 | 3.76 | 3.11 |

Table 3: Order of accuracy on irrregular grids with different $\nu$.

|  | Advection$\nu=10^{-8}, \operatorname{Re}=\sqrt{5} \times 10^{8}$ |  | Advection-Diffusion$\nu=1, \operatorname{Re}=\sqrt{5}$ |  | Diffusion$\nu=10^{8}, \operatorname{Re}=\sqrt{5} \times 10^{-8}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\varphi$ | $\mathrm{v}_{x}$ | $\varphi$ | $\mathrm{v}_{x}$ | $\varphi$ | $\mathrm{v}_{x}$ |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{1}\right)+\mathrm{DG}\left(\mathrm{P}_{0}\right)$ | 1.93 | 0.99 | 1.26 | 0.92 | 1.92 | 0.89 |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{2}\right)+\mathrm{DG}\left(\mathrm{P}_{1}\right)$ | 2.74 | 1.91 | 2.38 | 1.73 | 2.39 | 1.73 |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{3}\right)+\mathrm{DG}\left(\mathrm{P}_{2}\right)$ | 3.97 | 2.97 | 2.99 | 2.72 | 2.96 | 2.75 |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{2}\right)+$ rDG_LS $\left(\mathrm{P}_{0} \mathrm{P}_{1}\right)$ | 2.80 | 1.92 | - | - | - | - |
| DG $\left(\mathrm{P}_{0} \mathrm{P}_{3}\right)+$ rDG_LS $\left(\mathrm{P}_{1} \mathrm{P}_{2}\right)$ | 4.18 | 3.19 | 3.61 | 2.67 | 3.61 | 2.70 |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{2}\right)+\mathrm{rDG}-\mathrm{VR}\left(\mathrm{P}_{0} \mathrm{P}_{1}\right)$ | 2.74 | 1.93 | 2.78 | 2.21 | 2.81 | 2.22 |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{3}\right)+\mathrm{rDG}$-VR $\left(\mathrm{P}_{0} \mathrm{P}_{2}\right)$ | 3.77 | 2.97 | 3.02 | 2.62 | 3.01 | 2.78 |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{3}\right)+\mathrm{rDG}-\mathrm{VR}\left(\mathrm{P}_{1} \mathrm{P}_{2}\right)$ | 3.87 | 3.01 | 3.97 | 3.23 | 3.94 | 3.22 |

Overall, the hyperbolic rDG methods would be able to deliver the designed order of accuracy for most of the cases. However, we do observe that $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{2}\right)+\mathrm{rDG} \operatorname{LS}\left(\mathrm{P}_{0} \mathrm{P}_{1}\right)$ being unstable for non-advection limit case. This issue should be fixed by either adding more stencil to the reconstruction or applying limiters. Note that that $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{3}\right)+\mathrm{rDG} \mathrm{LS}\left(\mathrm{P}_{1} \mathrm{P}_{2}\right), \mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{3}\right)+\mathrm{rDG}-V R\left(\mathrm{P}_{0} \mathrm{P}_{2}\right)$, and $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{3}\right)+\mathrm{rDG}$-VR $\left(\mathrm{P}_{1} \mathrm{P}_{2}\right)$ are able to deliver 4 th order in $\varphi$ and 3rd order in gradients in all the cases very effectively. Meanwhile, for variational reconstruction, one can have global stencil with compact data structure, thus to resolve the stability issue and make the extension to higher order reconstruction more straightforward. Also, boundary condition can be ignored for using variational reconstruction. The numerical results indicate that the presented hyperbolic rDG schemes are attractive and worth further investigation.

Table 4: Order of accuracy on heterogeneous grids with different $\nu$.

|  | Advection$\nu=10^{-8}, \operatorname{Re}=\sqrt{5} \times 10^{8}$ |  | Advection-Diffusion$\nu=1, \operatorname{Re}=\sqrt{5}$ |  | Diffusion$\nu=10^{8}, \operatorname{Re}=\sqrt{5} \times 10^{-8}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\varphi$ | $\mathrm{v}_{x}$ | $\varphi$ | $\mathrm{v}_{x}$ | $\varphi$ | $\mathrm{v}_{x}$ |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{1}\right)+\mathrm{DG}\left(\mathrm{P}_{0}\right)$ | 2.11 | 0.95 | 1.17 | 1.00 | 1.98 | 1.00 |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{2}\right)+\mathrm{DG}\left(\mathrm{P}_{1}\right)$ | 3.15 | 2.07 | 2.60 | 1.92 | 2.60 | 1.92 |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{3}\right)+\mathrm{DG}\left(\mathrm{P}_{2}\right)$ | 4.08 | 3.04 | 3.12 | 2.76 | 3.10 | 2.76 |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P} 2\right)+\mathrm{rDG}$ _LS $\left(\mathrm{P}_{0} \mathrm{P}_{1}\right)$ | 3.05 | 2.06 | - | - | - | - |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{3}\right)+\mathrm{rDG}$ LS $\left(\mathrm{P}_{1} \mathrm{P}_{2}\right)$ | 4.17 | 3.11 | 3.87 | 2.99 | 4.00 | 3.10 |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{2}\right)+\mathrm{rDG}-\mathrm{VR}\left(\mathrm{P}_{0} \mathrm{P}_{1}\right)$ | 3.05 | 2.06 | 2.38 | 2.01 | 2.39 | 2.21 |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{3}\right)+$ rDG_VR( $\left.\mathrm{P}_{0} \mathrm{P}_{2}\right)$ | 4.51 | 3.27 | 3.57 | 3.09 | 3.53 | 3.04 |
| $\mathrm{DG}\left(\mathrm{P}_{0} \mathrm{P}_{3}\right)+\mathrm{rDG}-\mathrm{VR}\left(\mathrm{P}_{1} \mathrm{P}_{2}\right)$ | 4.23 | 3.08 | 3.89 | 3.11 | 3.86 | 2.99 |



Figure 5: Grid refinement study on regular grids with $\nu=10^{-8}, \operatorname{Re}=\sqrt{5} \times 10^{8}$.


Figure 6: Grid refinement study on regular grids with $\nu=1, \operatorname{Re}=\sqrt{5}$.


Figure 7: Grid refinement study on regular grids with $\nu=10^{8}, \operatorname{Re}=\sqrt{5} \times 10^{-8}$.


Figure 8: Grid refinement study on irregular grids with $\nu=10^{-8}, \operatorname{Re}=\sqrt{5} \times 10^{8}$.


Figure 9: Grid refinement study on irregular grids with $\nu=1, \operatorname{Re}=\sqrt{5}$.


Figure 10: Grid refinement study on irregular grids with $\nu=10^{8}, \operatorname{Re}=\sqrt{5} \times 10^{-8}$.


Figure 11: Grid refinement study on heterogeneous grids with $\nu=10^{-8}, \operatorname{Re}=\sqrt{5} \times 10^{8}$.


Figure 12: Grid refinement study on heterogeneous grids with $\nu=1, \operatorname{Re}=\sqrt{5}$.


Figure 13: Grid refinement study on heterogeneous grids with $\nu=10^{8}, \operatorname{Re}=\sqrt{5} \times 10^{-8}$.

## V. Conclusions and Outlook

High order reconstructed discontinuous Galerkin (rDG) methods based on first-order hyperbolic system (FOHS) for advection-diffusion equations have been developed and presented in the study. With FOHS formulation, an equivalent hyperbolic system, which would yield at the same steady solution, is generated. The rDG method, naturally designed for hyperbolic system, could provide the high-order solutions for both primary variables and its gradients efficiently. The numerical examples showed in the paper illustrate the capability and the potential of the developed methods, indicating that the hyperbolic rDG methods provide attractive alternatives to solve advection-diffusion equations. Future work would be focused on extending the hyperbolic rDG method to Navier-Stokes equation on fully 3D unstructured grids.

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[^1][^2]
[^0]:    *Ph.D Student, Department of Mechanical and Aerospace Engineering, Student Member AIAA.
    ${ }^{\dagger}$ Ph.D Student, Department of Mechanical and Aerospace Engineering, Student Member AIAA.
    ${ }^{\ddagger}$ Postdoctoral Research Associate, Department of Mechanical and Aerospace Engineering.
    §Professor, Department of Mechanical and Aerospace Engineering, Associate Fellow AIAA.
    «Associate Research Fellow, National Institute of Aerospace, Associate Fellow AIAA.

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