Grids and Solutions from Residual Minimization

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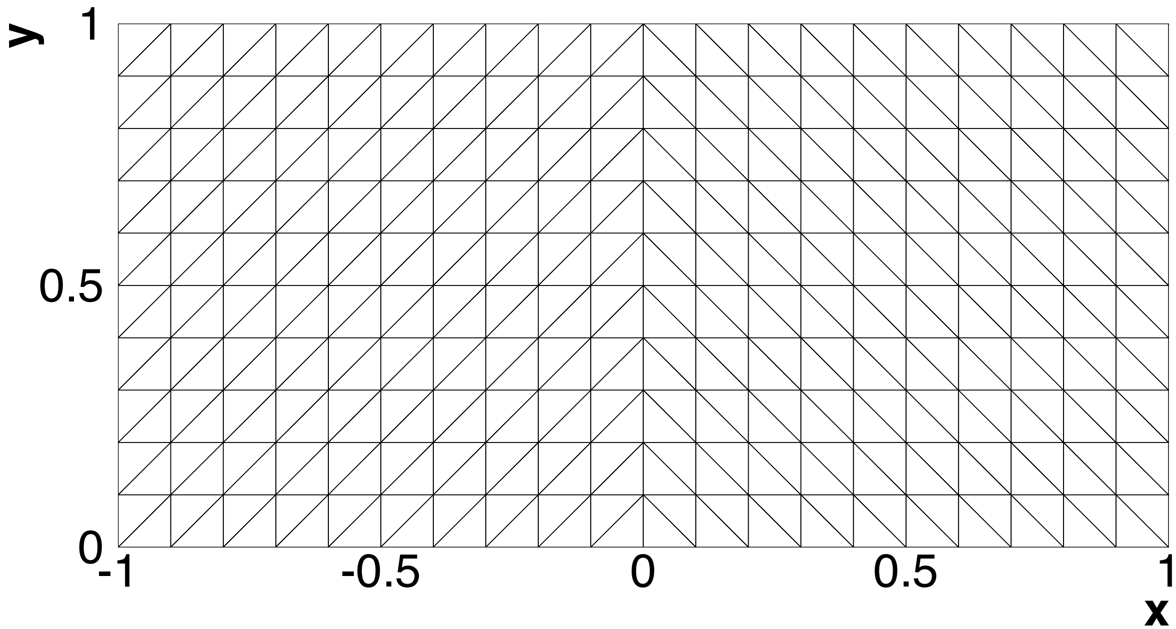
Circular Advection

$$y\,\partial_x u - x\,\partial_y u = 0$$



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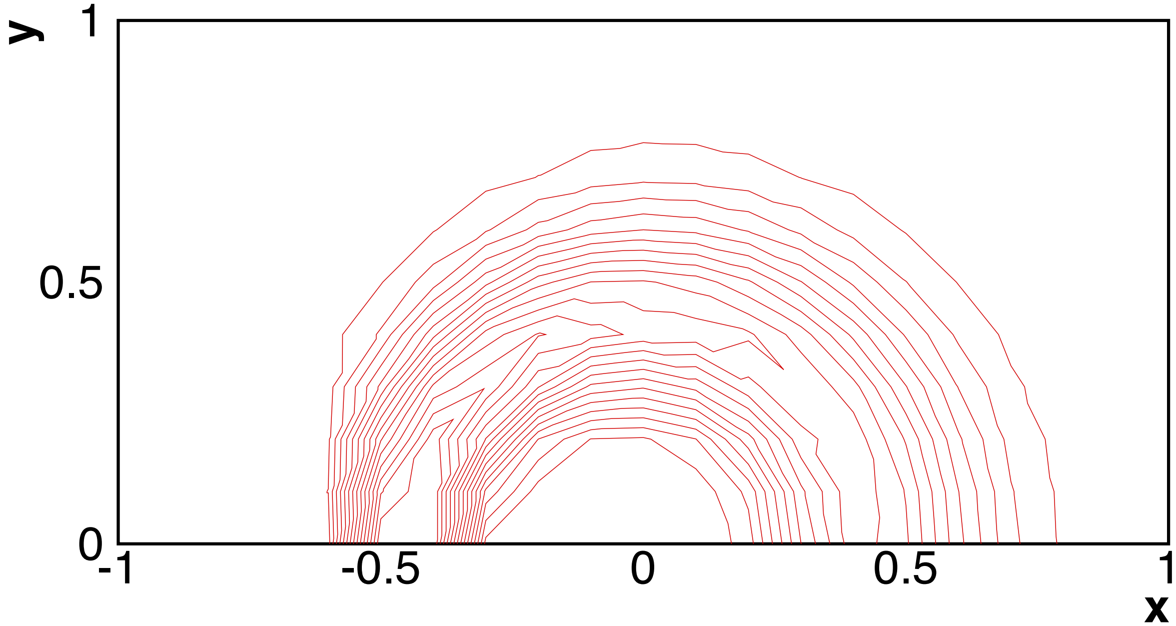
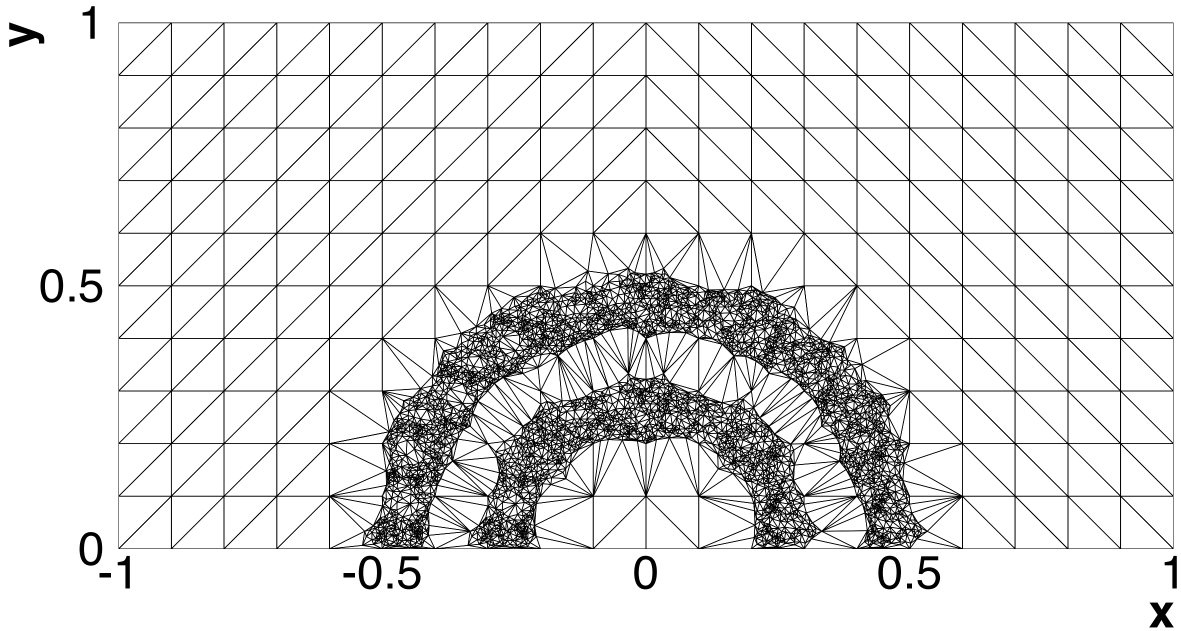
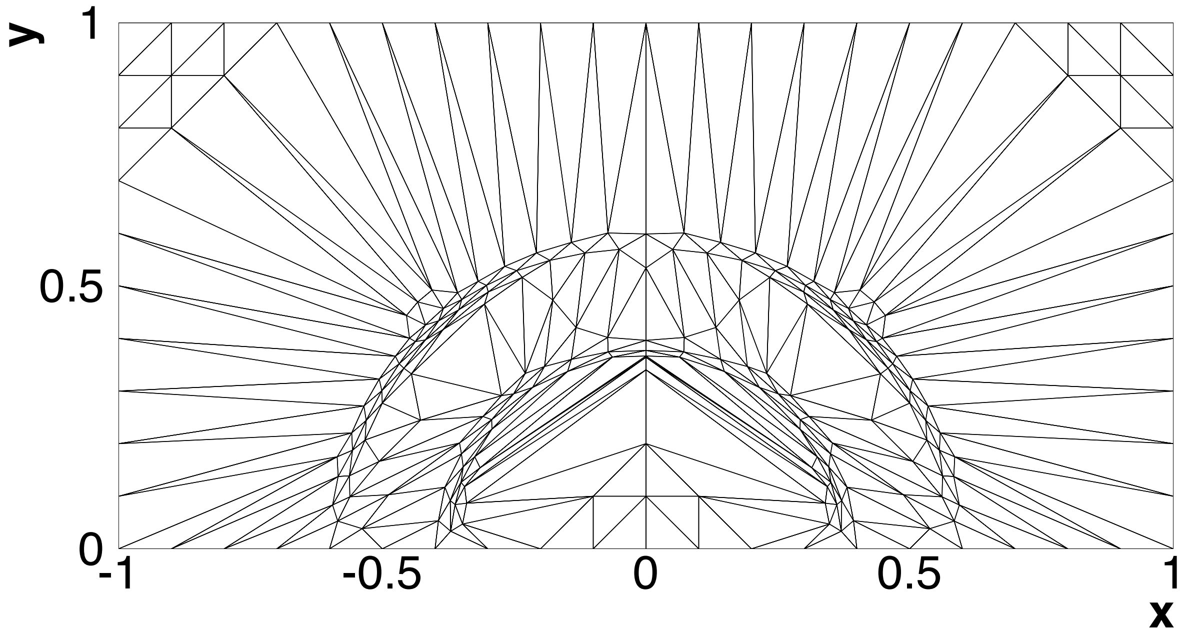


A typical solution on the regular grid.

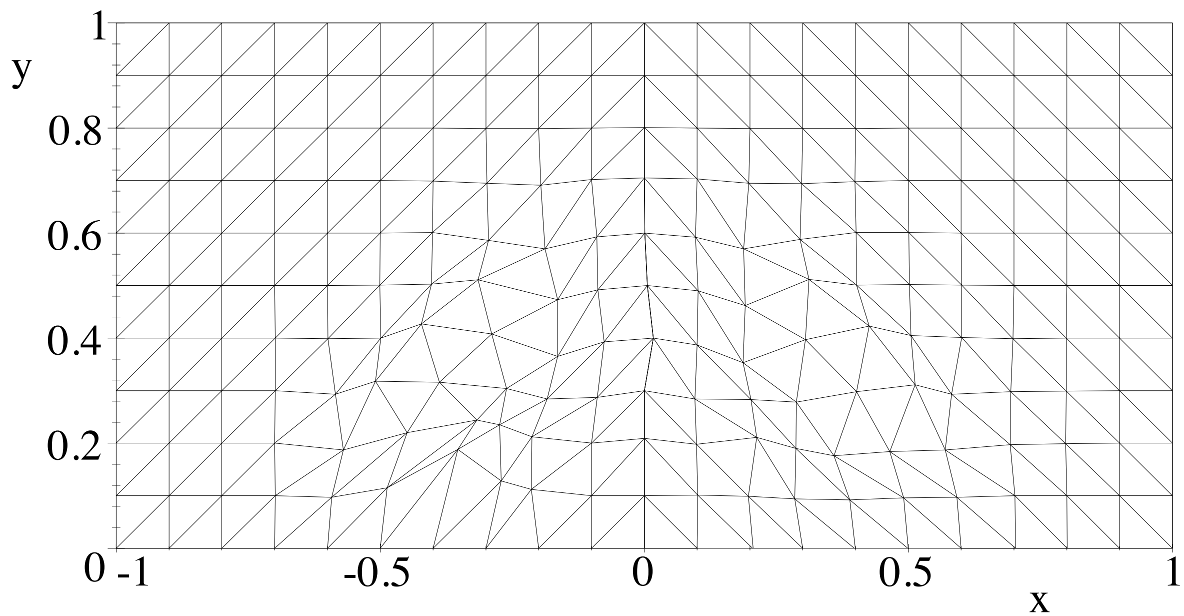
Mesh Refinement



Mesh Movement



An Adaptive Grid



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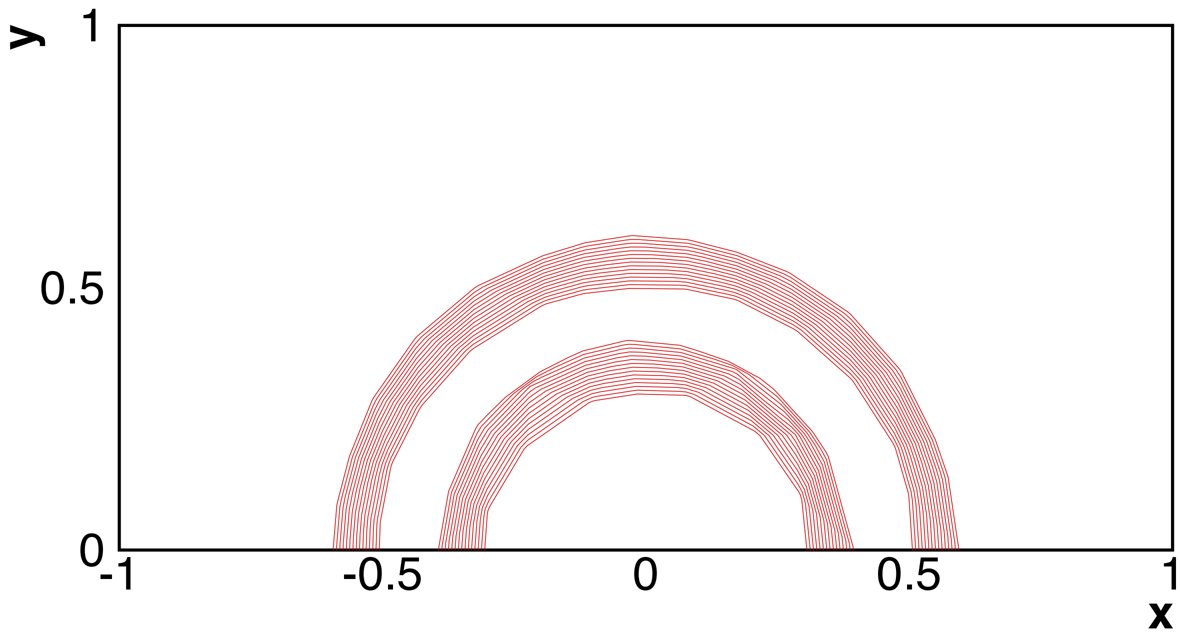
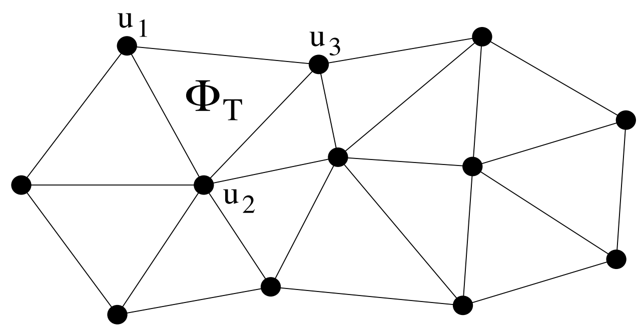


Why Triangular Grids?

- Easy to Create for Complex Domains
- Easy to Insert/Delete Nodes
- Easy to Change the Connectivity (Edge Swapping)
- *Additional Degrees of Freedom*

$$\boxed{\text{The Number of Elements} \approx 2 \times \text{The Number of Nodes}}$$

Least-Squares Residual Minimization



FLUCTUATION:

$$\mathbf{\Phi}_T = \int_T [\partial_x \mathbf{F} + \partial_y \mathbf{G}] \, dx \, dy$$

NORM:

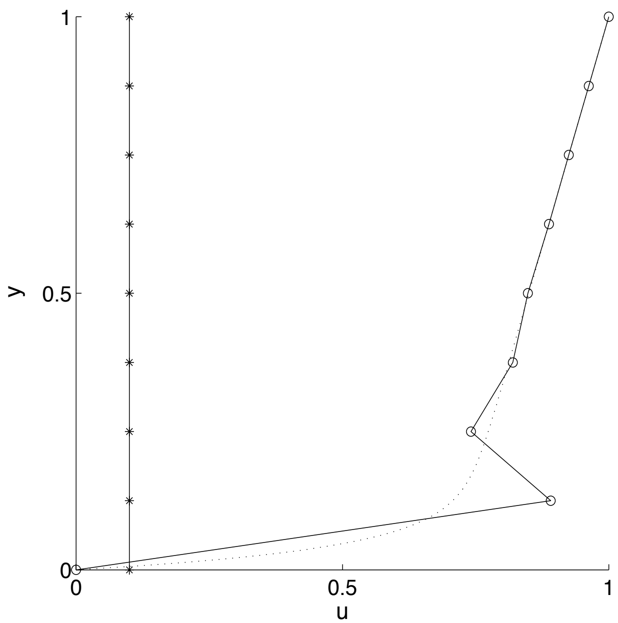
$$\mathcal{F} = \frac{1}{2} \sum \mathbf{\Phi}_T^t Q_T \mathbf{\Phi}_T$$

UPDATES:

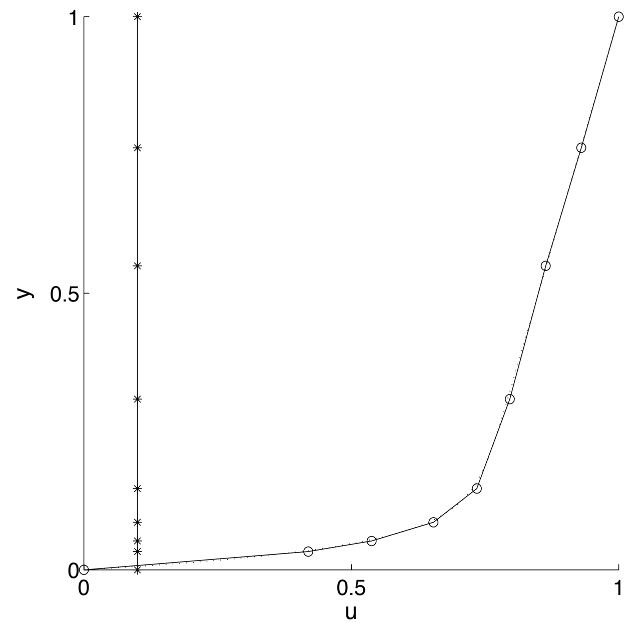
$$\delta \mathbf{u}_j = -\,\omega_u \frac{\partial \mathcal{F}}{\partial \mathbf{u}_j}, \quad \delta \mathbf{x}_j = -\,\omega_x \frac{\partial \mathcal{F}}{\partial \mathbf{x}_j}$$

Example 1: One-Dimensional Problem

$$\epsilon \frac{d^2u}{dy^2} + \frac{du}{dy} - a = 0$$



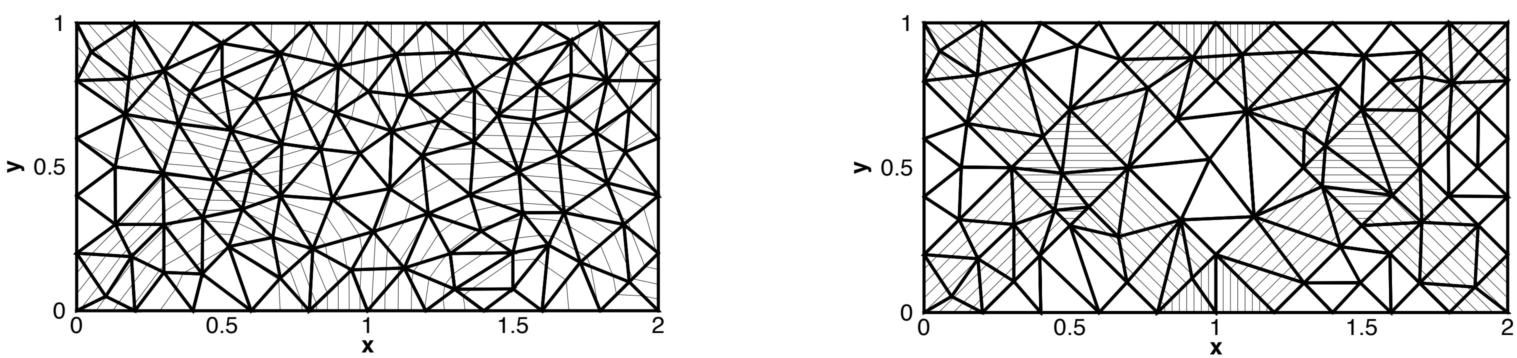
A Fixed Uniform Grid



An Adaptive Grid

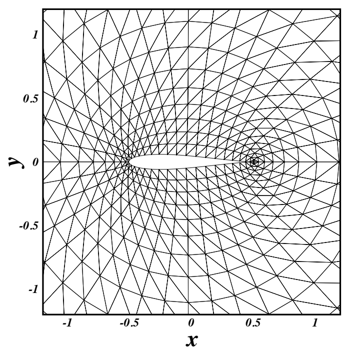
Example 2: Hyperbolic Problem

$$(1 - M_\infty^2)\partial_x u + \partial_y v = 0, \quad \partial_x v - \partial_y u = 0$$

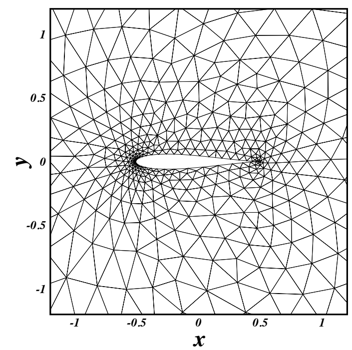


Supersonic flow through a duct, $M_\infty = \sqrt{2}$ with a small incidence.

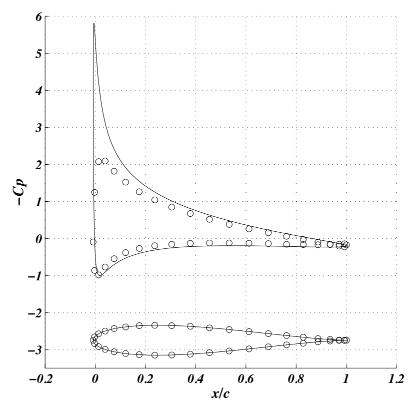
Example 3: Elliptic Problem



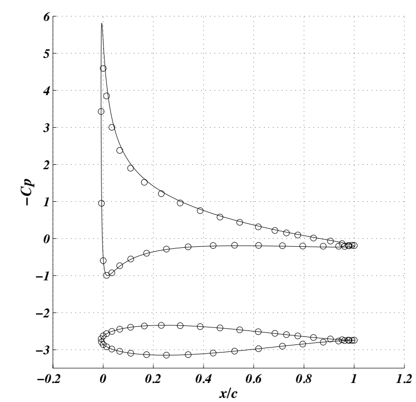
Regular 40x20 Grid



An Adaptive Grid



C_p Distribution



C_p Distribution

Linear Advection

LINEAR ADVECTION EQUATION:

$$a\,\partial_x u + b\,\partial_y u = 0$$

The solution u is convected in the direction (a, b).

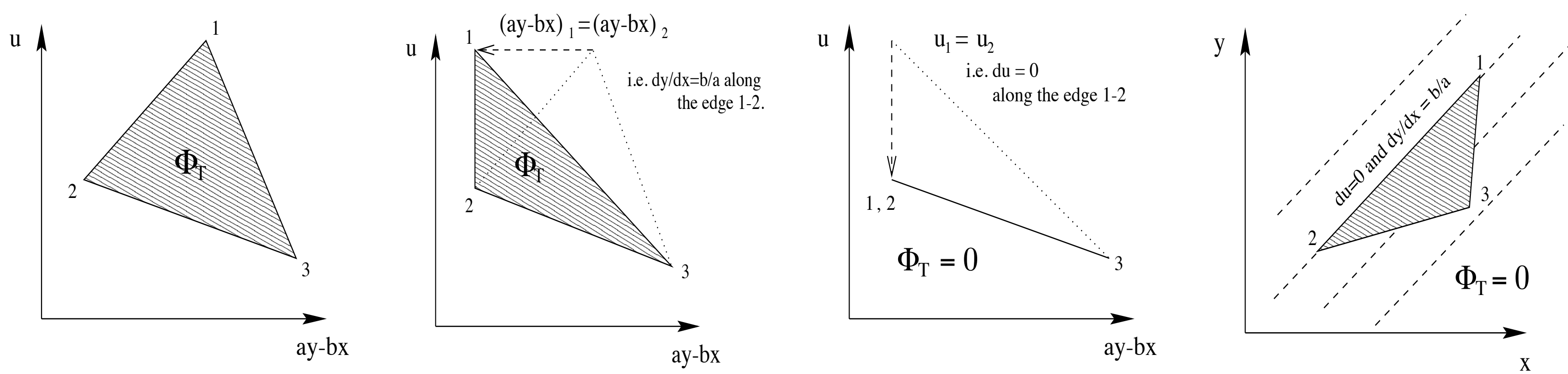
CHARACTERISTIC EQUATION:

$$u = const. \quad \text{along} \quad dy/dx = b/a$$

Fluctuation

$$\Phi_T = \int_T [a\,\partial_x u + b\,\partial_y u]\; dx\,dy = \frac{1}{2}\sum_{i\in j_T} u_i\,(a\Delta y_i - b\Delta x_i)$$

The fluctuation vanishes if the characteristic equation is satisfied along one edge, and it is independent of the third node.



Least-Squares Minimization

NORM:

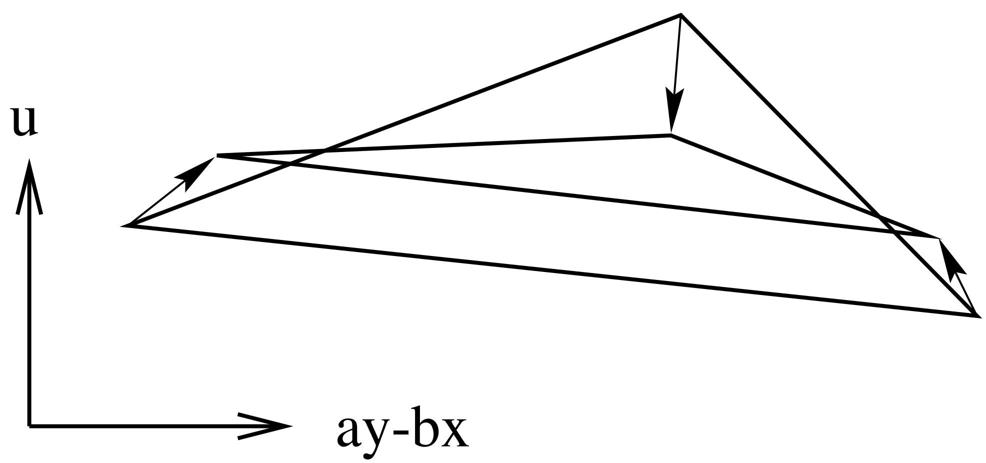
$$\mathcal{F} = \sum F_T = \frac{1}{2} \sum \frac{\mathbf{\Phi}_T^2}{S_T}$$

UPDATES:

$$\begin{aligned}
\delta u_j &= -c_u \frac{\partial \mathcal{F}}{\partial u_j} = \frac{c_u}{2} \sum_{\{T_j\}} (a\Delta y_T - b\,\Delta x_T) \frac{\Phi_T}{S_T} \\
\delta \mathbf{x}_j &= -c_x \frac{\partial \mathcal{F}}{\partial \mathbf{x}_j} = \sum_{\{T_j\}} \frac{c_x}{2} \Delta u_T \frac{\Phi_T}{S_T} \begin{bmatrix} -b \\ a \end{bmatrix} + c_x \sum_{\{T_j\}} \frac{F_T}{2S_T} \begin{bmatrix} \Delta y_T \\ -\Delta x_T \end{bmatrix}
\end{aligned}$$

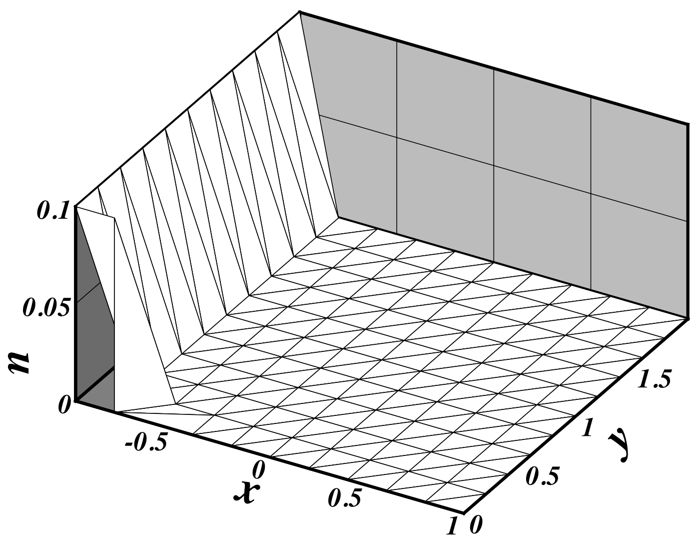
The node moves in the direction normal to the characteristic.

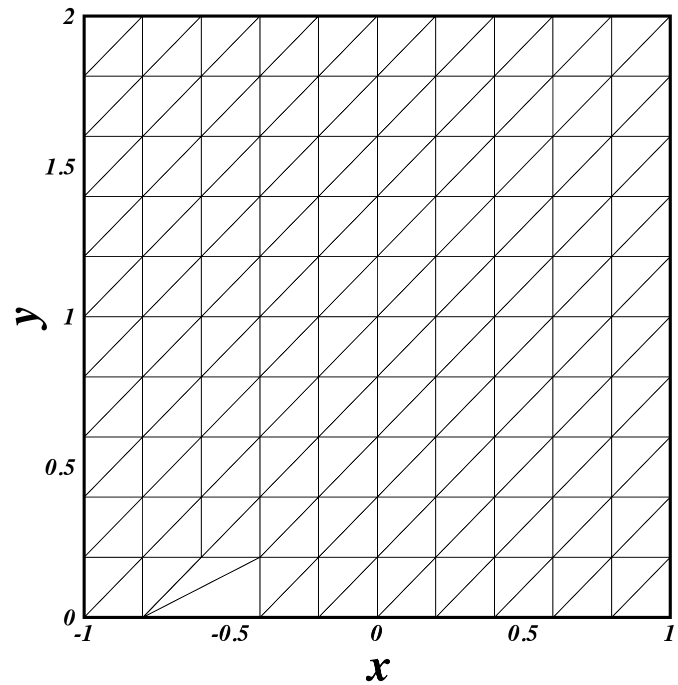
Geometrical View



Minimizing the area of the triangle in the characteristic plane as quickly as possible.

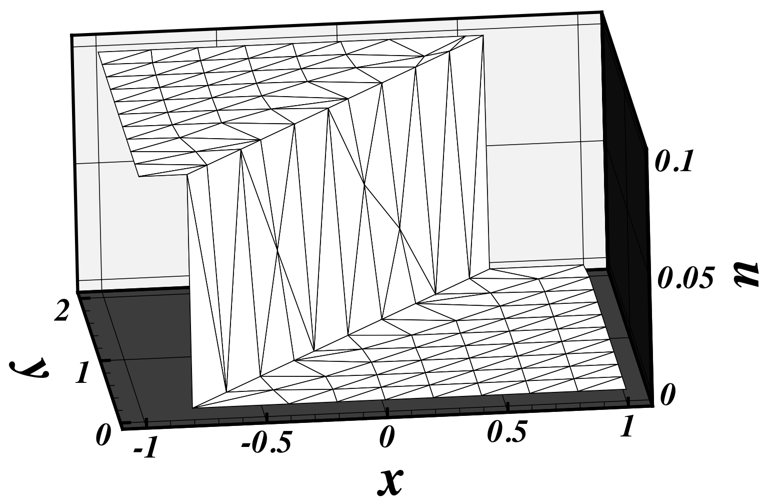
A Simple Linear Advection

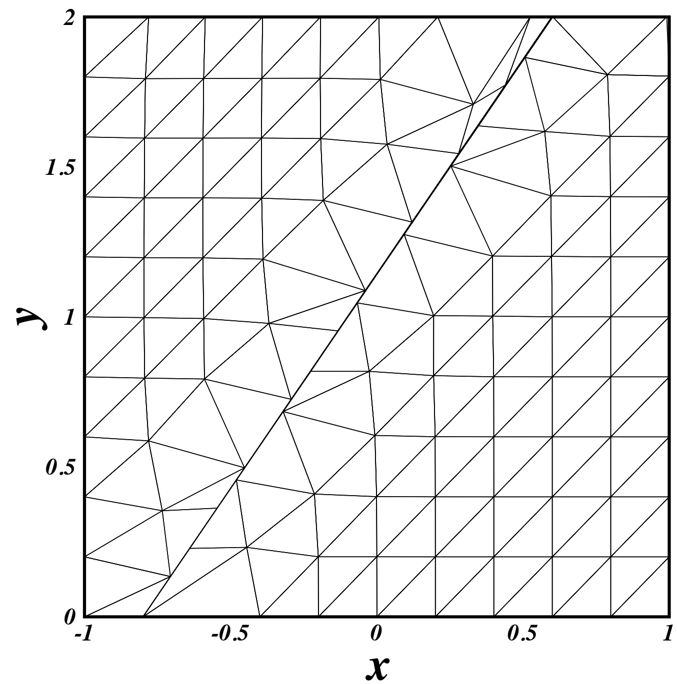




A degenerate element is introduced to represent a perfect discontinuity.

A Simple Linear Advection





The grid is altered only in an important region.

A Linear Hyperbolic System

Small Perturbation Aerodynamics:

$$(1 - M^2)\partial_x u + \partial_y v = 0,\ \ \partial_x v - \partial_y u = 0$$

Characteristic Equations (supersonic case):

$$\begin{aligned} \beta u + v &= const. \quad \text{along } dy/dx = -1/\beta \\ \beta u - v &= const. \quad \text{along } dy/dx = 1/\beta \end{aligned}$$

where $\beta = \sqrt{M^2 - 1}$.

Fluctuations

FLUCTUATIONS:

$$\Delta_T = \int_T \left[\beta^2 \partial_x u - \partial_y v\right] \, dx\, dy, \;\; \Omega_T = \int_T \left[\partial_x v - \partial_y u\right] \, dx\, dy$$

CHARACTERISTIC FLUCTUATIONS:

$$\begin{aligned} C_T &= \frac{1}{2} \sum_{i \in j_T} (v + \beta u)_i \Delta (x + \beta y)_i = \Delta_T + \beta \Omega_T \\ D_T &= \frac{1}{2} \sum_{i \in j_T} (v - \beta u)_i \Delta (x - \beta y)_i = \Delta_T - \beta \Omega_T \end{aligned}$$

The Least-Squares Norm

Minimize the characteristic fluctuations in the least-squares norm.

$$\mathcal{F} = \sum_{T \in \{T\}} F_T = \frac{1}{2} \sum_{T \in \{T\}} \frac{C_T^2 + D_T^2}{S_T}$$

In terms of the original fluctuations.

$$\mathcal{F} = \frac{1}{2} \sum_{T \in \{T\}} \frac{\Delta_T^2 + \beta^2 \Omega_T^2}{S_T}$$

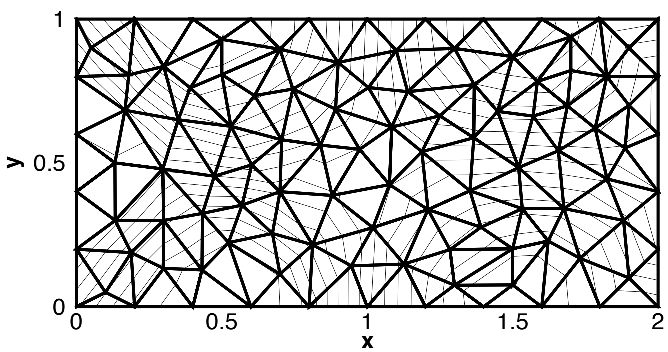
In the matrix form,

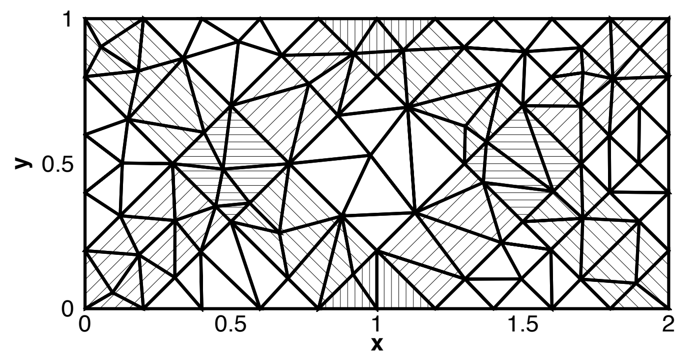
$$\mathcal{F} = \frac{1}{2} \sum \frac{\mathbf{\Phi}_T^t Q_T \mathbf{\Phi}_T}{S_T}$$

where

$$\mathbf{\Phi}_T = (\Delta_T, \Omega_T)^t \,, \ Q_T = \begin{bmatrix} 1 & 0 \\ 0 & \beta^2 \end{bmatrix}$$

Small Perturbation Aerodynamics

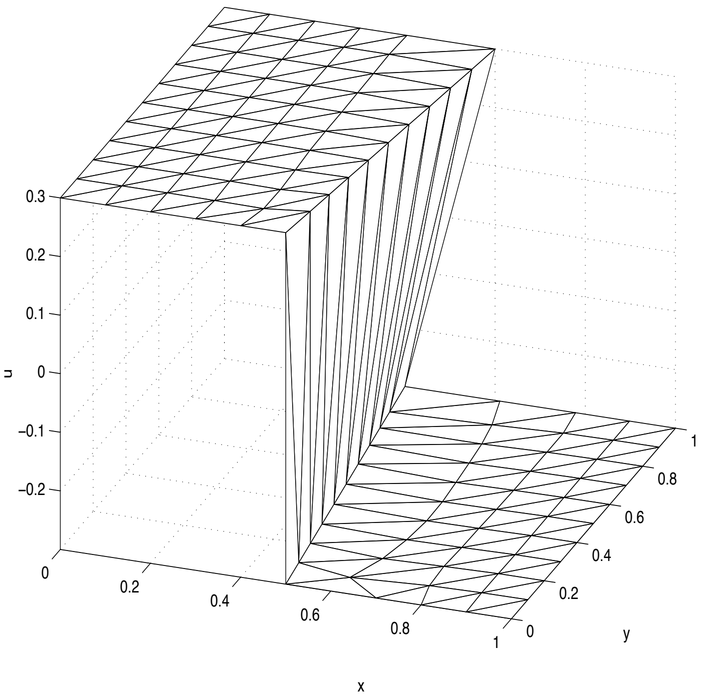


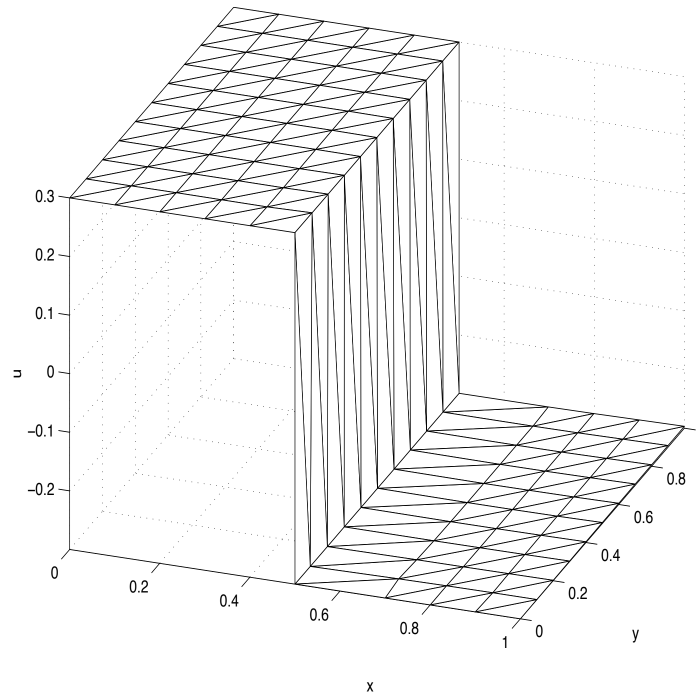


Supersonic flow through a duct, $M_\infty = \sqrt{2}$ with a small incidence. 18 nodes have been removed on the final grid.

Burgers' Equation

$$\partial_y u + u\,\partial_x u = 0$$





$$\Phi_T = \frac{1}{2}\sum_{i\in j_T} u_i\,(\Delta y_i - \bar{u}_T \Delta x_i) \quad \Phi_T = \frac{1}{2}\sum_{i\in j_T} u_i\,(\Delta y_i - u_i \Delta x_i)$$

Quadrature Formulae

Conservation Laws:

$$\partial_x \mathbf{F}(\mathbf{w}) + \partial_y \mathbf{G}(\mathbf{w}) = 0$$

Bilinear Flux Functions:

$$\mathbf{F}(\mathbf{w}) = \mathbf{w}^t \mathbf{C} \mathbf{w}, \qquad \mathbf{G}(\mathbf{w}) = \mathbf{w}^t \mathbf{D} \mathbf{w}$$

Fluctuation:

$$\boldsymbol{\Phi}_{123} = \int\int_{123} \left[\partial_x \mathbf{F}(\mathbf{w}) + \partial_y \mathbf{G}(\mathbf{w})\right] dx dy = \oint_{123} \left[\mathbf{F}(\mathbf{w}) dy + \mathbf{G}(\mathbf{w}) dx\right]$$

Quadrature Formulae:

$$\int_1^2 \mathbf{F}(\mathbf{w}) dy = (y_2 - y_1) \left[\mathbf{w}_1^t \mathbf{C} \mathbf{w}_2 + \frac{\alpha}{2} (\mathbf{w}_1 - \mathbf{w}_2) \mathbf{C} (\mathbf{w}_1 - \mathbf{w}_2)\right]$$

Formulae for Fluctuation

$$\begin{aligned}\mathbf{\Phi}_{123}(\alpha) &= \mathbf{w}_{31}^t(\alpha)\left\{\Delta y_3\mathbf{C} - \Delta x_3\mathbf{D}\right\}(\mathbf{w}_3 - \mathbf{w}_1) + \\ &\quad \mathbf{w}_{12}^t(\alpha)\left\{\Delta y_2\mathbf{C} - \Delta x_2\mathbf{D}\right\}(\mathbf{w}_2 - \mathbf{w}_1)\end{aligned}$$

where

$$\mathbf{w}_{ij}(\alpha) = \frac{\alpha}{2}(\mathbf{w}_i + \mathbf{w}_j) + (1-\alpha)\mathbf{w}_k.$$

Or written

$$\begin{aligned}\mathbf{\Phi}_{123}(\alpha) &= \left\{\frac{\partial \mathbf{F}}{\partial \mathbf{w}}\Delta y_3 - \frac{\partial \mathbf{G}}{\partial \mathbf{w}}\Delta x_3\right\}(\mathbf{w}_3 - \mathbf{w}_1) + \\ &\quad \left\{\frac{\partial \mathbf{F}}{\partial \mathbf{w}}\Delta y_2 - \frac{\partial \mathbf{G}}{\partial \mathbf{w}}\Delta x_2\right\}(\mathbf{w}_2 - \mathbf{w}_1)\end{aligned}$$

(Note that there are two other rearrangements possible)

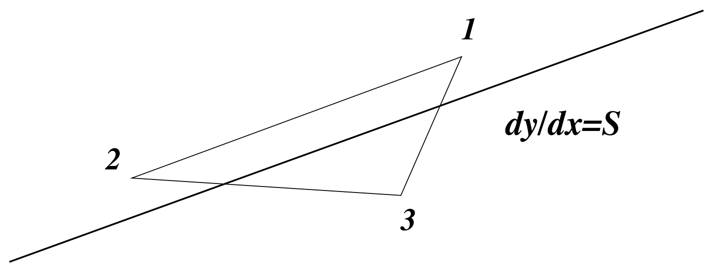
Shock Recognition: $\alpha = 1$

If $\mathbf{w}_1 = \mathbf{w}_2 = \mathbf{w}_c$,

$$\boldsymbol{\Phi}_{123}(\alpha) = \mathbf{w}_{31}^t(\alpha)\left\{\Delta y_3 \mathbf{C} - \Delta x_3 \mathbf{D}\right\}(\mathbf{w}_3 - \mathbf{w}_c).$$

Take $\alpha = 1 \Longrightarrow \mathbf{w}_{31}(\alpha) = \overline{\mathbf{w}}_{31} = (\mathbf{w}_3 + \mathbf{w}_c)/2.$
This corresponds precisely with the Hugoniot condition,

$$\overline{\mathbf{w}}^t(S\mathbf{C} - \mathbf{D})\Delta\mathbf{w} = 0$$



$\boldsymbol{\Phi}_{123}(1)$ vanishes if the Hugoniot condition is satisfied across a shock *and "one edge is aligned with the shock".*

Entropy-Satisfying Solution: $\alpha = 0$

For Burgers' equation $\partial_t u + \partial_x f = 0$ where $f = u^2/2$,

$$\frac{\partial u_j}{\partial t} + \frac{f_{j+1} - f_{j-1}}{2\Delta x} = (1-\alpha)\Delta x^2 \frac{u_{j+1} - u_{j-1}}{2\Delta x} \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2}$$

which is a second order discretisation of

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = (1-\alpha)\Delta x^2 \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2}.$$

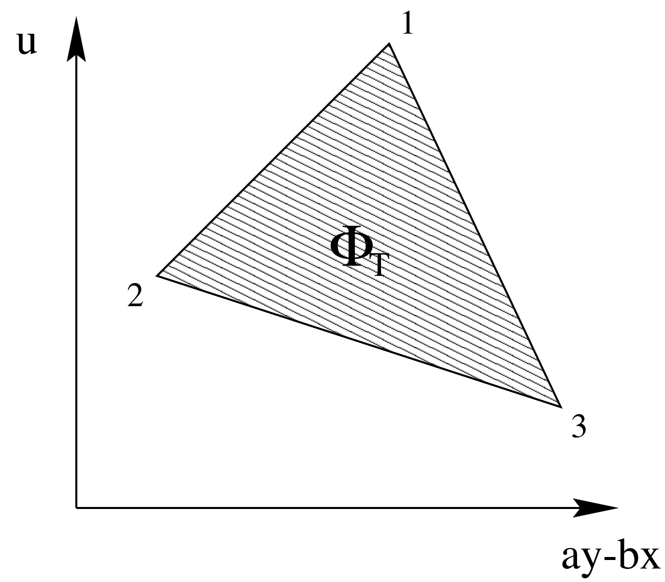
In the case of expansion, $\frac{\partial u}{\partial x} > 0$,
$\alpha < 1$ produces positive dissipation —> physical solutions.

Characteristic Recognition: $\alpha = 0$

If $\mathbf{w}_1 = \mathbf{w}_2 = \mathbf{w}_c$,

$$\mathbf{\Phi}_{123}(\alpha) = \mathbf{w}_{31}^t(\alpha)\left\{\Delta y_3 \mathbf{C} - \Delta x_3 \mathbf{D}\right\}(\mathbf{w}_3 - \mathbf{w}_c).$$

For the special choice $\alpha = 0 \Longrightarrow \mathbf{w}_{31}(\alpha) = \mathbf{w}_2$.



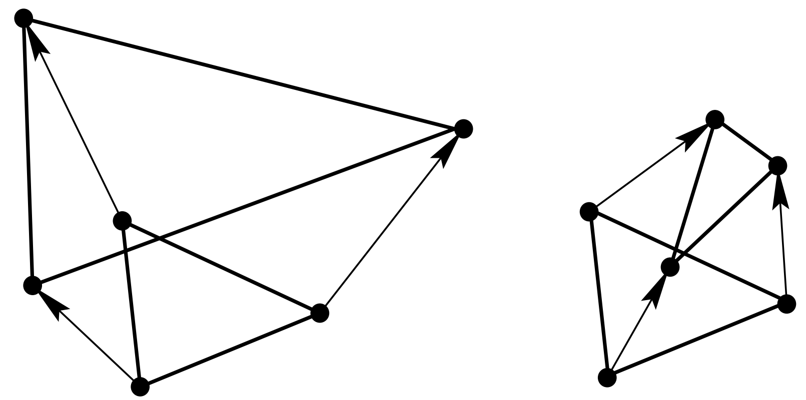
For scalar problems, $\mathbf{\Phi}_{123}(0)$ vanishes if the characteristic

0.5pt=0.482150.5pt

equation $d\mathbf{w} = 0$ **is exactly satisfied along one edge**

and "that edge is aligned with the exact characteristic".

Detecting Compression/Expansion



Left: Element in expansion. Right: Element in compression.
Arrows indicate the characteristic speed vecors.

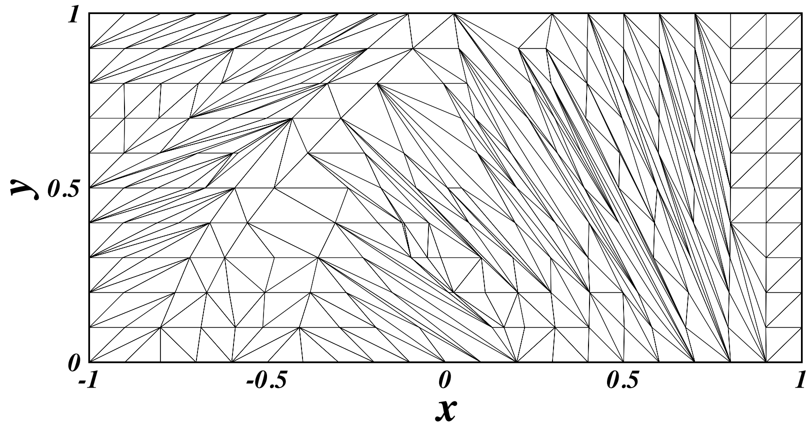
Useful quantity:

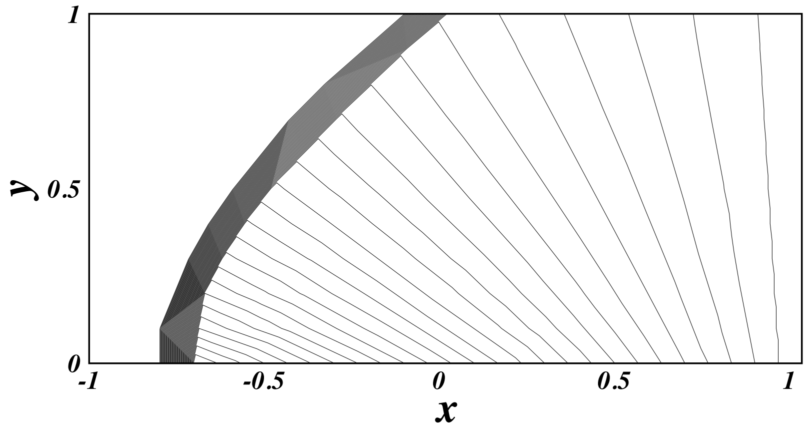
$$\frac{dS_T}{dt} = \frac{1}{2} \sum_{i=1,2,3} \lambda_i \cdot \mathbf{n}_i$$

Solution Procedure

1. Converge the solution on a fixed grid
2. Assign α to each element, depending on the sign of dS_T/dt
3. Remove undesirable nodes
4. Update solutions and coordinates, with edge swappings interleaved.

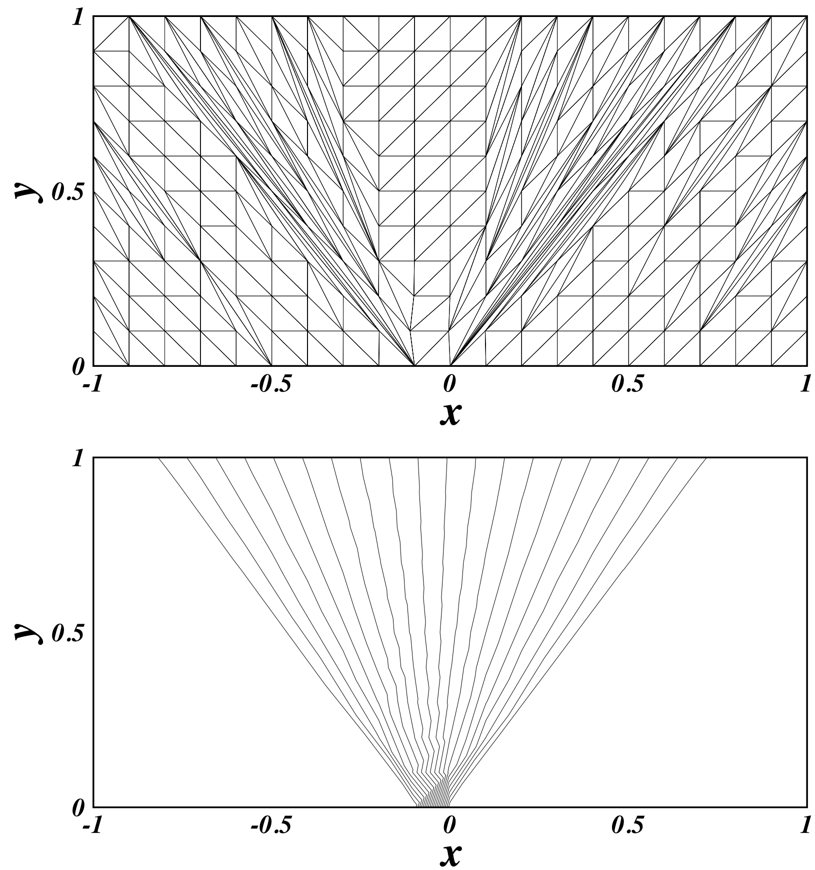
A Curved Shock





Grid and Solution for a curved shock problem

A Grid-Aligned Rarefaction



Grid and Solution for a rarefaction

Cauchy-Riemann System

$$
\begin{aligned}
\partial_x \phi - \partial_y \psi &= 0 \\
\partial_y \phi + \partial_x \psi &= 0
\end{aligned}
$$

.

Two-dimensional incompressible potential flows with velocity potential ϕ, stream function ψ. But all 2D elliptic systems are isomorphic with the Cauchy-Riemann system.

Least-Squares Minimization

FLUCTUATION:

$$U_T = \int_T \left[\partial_x \phi - \partial_y \psi\right] \; dx\,dy = \int_T \left[\partial_\psi y - \partial_\phi x\right] \; d\phi\,d\psi$$

$$V_T = \int_T \left[\partial_y \phi + \partial_x \psi\right] \; dx\,dy = \int_T \left[-\partial_\phi y - \partial_\psi x\right] \; d\phi\,d\psi$$

NORM:

$$\mathcal{F} = \sum_{T\in\{T\}} F_T = \frac{1}{2} \sum_{T\in\{T\}} \frac{U_T^2 + V_T^2}{S_T}$$

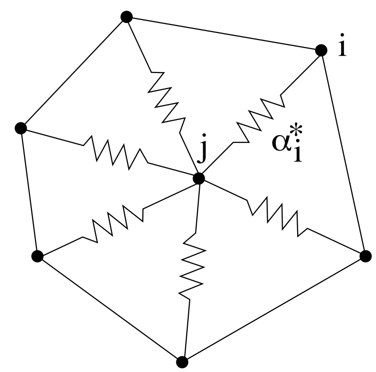
Equivalent to minimizing

$$\mathcal{F} = \frac{1}{2} \sum_{T\in\{T\}} \iint_T \nabla\phi\cdot\nabla\phi\, dxdy + \frac{1}{2} \sum_{T\in\{T\}} \iint_T \nabla\psi\cdot\nabla\psi\, dxdy - \sum_{T\in\{T\}} S_T'$$

Equivalent to the standard finite element method applied to

$$\phi_{xx} + \phi_{yy} = 0, \quad \psi_{xx} + \psi_{yy} = 0 \text{ with } \psi = g(x,y),\; \partial_n \phi = \partial_s \psi$$

Mesh Movement (Spring Analogy)



The equations we need to solve, $\frac{\partial \mathcal{F}}{\partial \mathbf{x}_j} = 0$, can be written in the interative form.

$$\mathbf{x}_j^{n+1} = \mathbf{x}_j^n + \frac{\sum_{i \in i_j} \alpha_i^* (\mathbf{x}_i^n - \mathbf{x}_j^n)}{\sum_{i \in i_j} \alpha_i^*} + \frac{1}{2} \frac{\sum_{T \in \{T_j\}} F_T / S_T}{\sum_{i \in i_j} \alpha_i^*} \mathbf{n}_T$$

where

$$\alpha_i^* = \frac{1}{4}\left(|J_T| \cot\theta_T + |J_{T+1}| \cot\theta_{T+1}\right)$$

$$J_T = (\partial_x \phi \partial_y \psi - \partial_y \phi \partial_x \psi)|_T$$

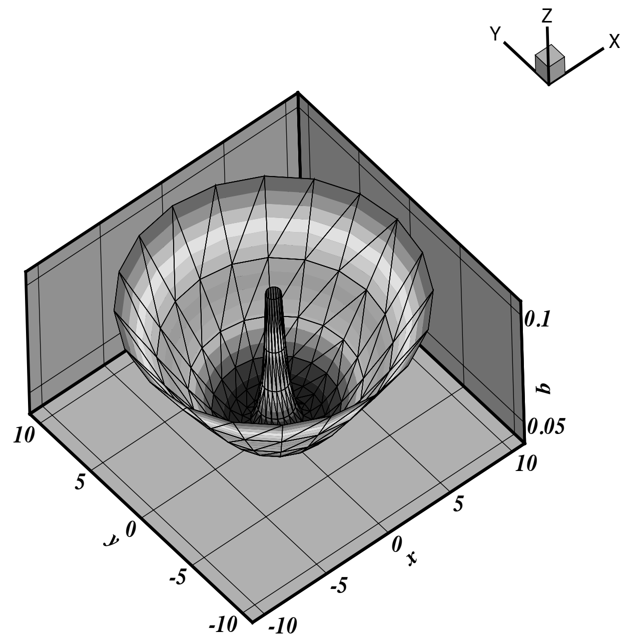
Cauchy-Riemann System

$$
\begin{aligned}
\partial_x u + \partial_y v &= 0 \\
\partial_x v - \partial_y u &= 0
\end{aligned}
$$

.

"Solutions are continuous in doubly-connected regions."

A Potential Vortex: A Test Problem



The exact solution is $q = 1/r$. But the Laplace equations allow another solution.

$$q = 1/r + r$$

Recovering the Second-Order Accuracy

Minimize the "unweighted norm".

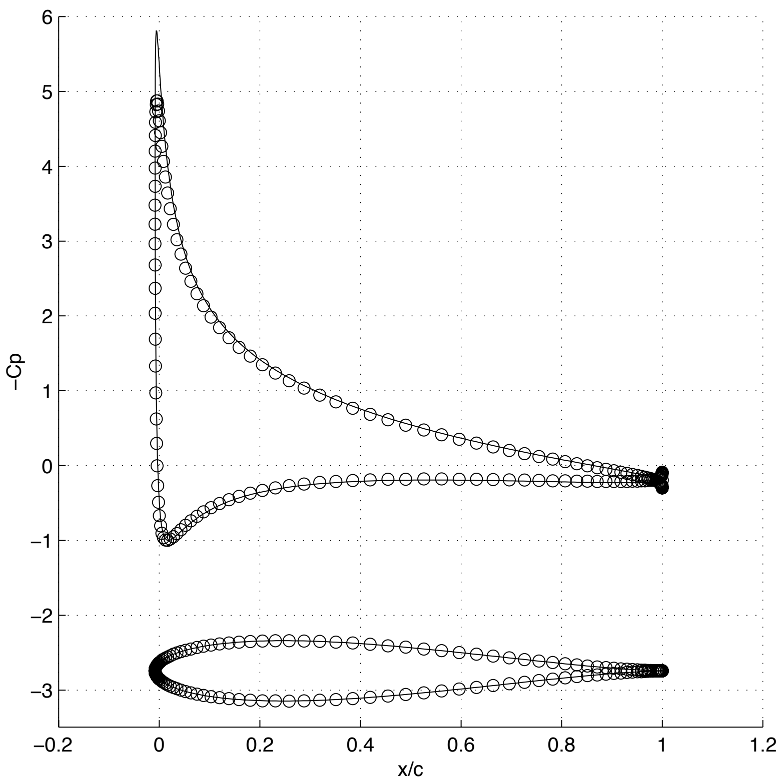
$$\mathcal{F} = \sum_{T \in \{T\}} F_T = \frac{1}{2} \sum_{T \in \{T\}} \left[U_T^2 + V_T^2 \right]$$

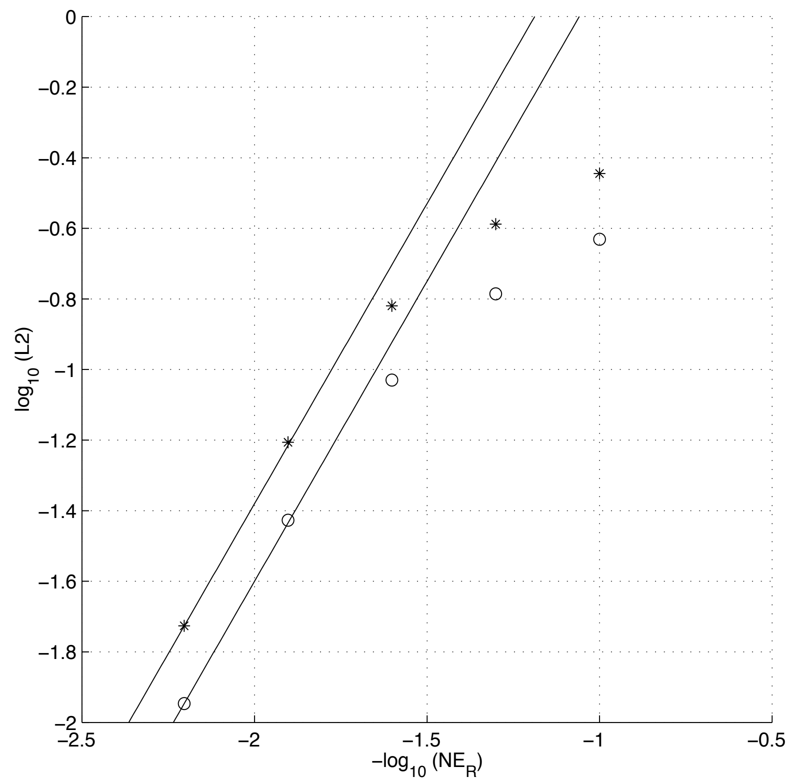
Equivalent to minimizing

$$\mathcal{F} = \frac{1}{2} \sum_{T \in \{T\}} S_T \iint_T \nabla\phi \cdot \nabla\phi \, dx dy + \frac{1}{2} \sum_{T \in \{T\}} S_T \iint_T \nabla\psi \cdot \nabla\psi \, dx dy - \sum_{T \in \{T\}} S_T S_T'$$

"The last term creates the variable-coupling, and it is no longer equivalent to the FEM for the Laplace equations."

A Solution by the Unweighted Norm





Solution obtained on an 160x80 O-grid

The convergence rate is 1.7

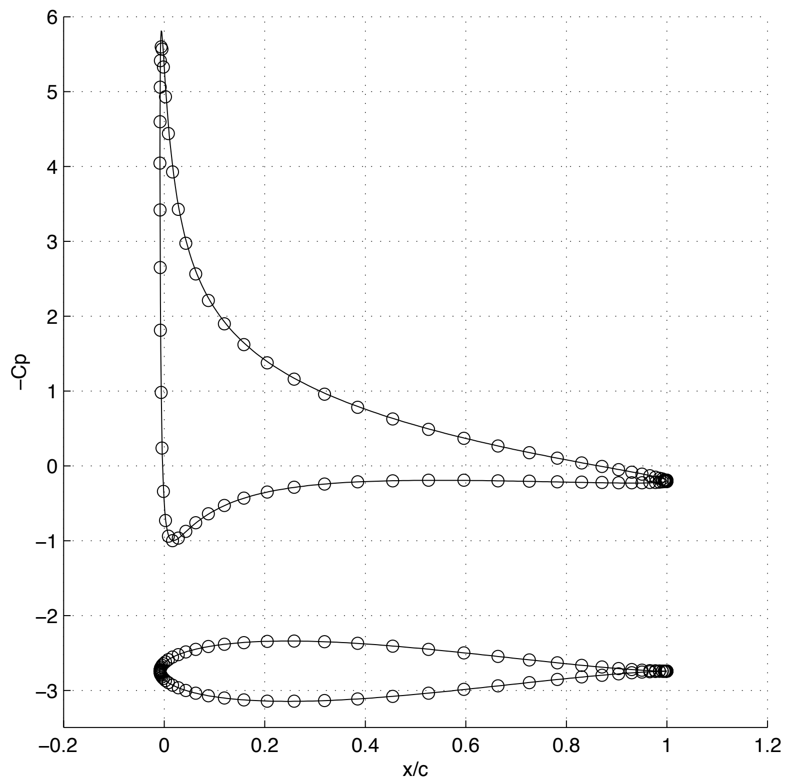
A Third-Order Least-Squares

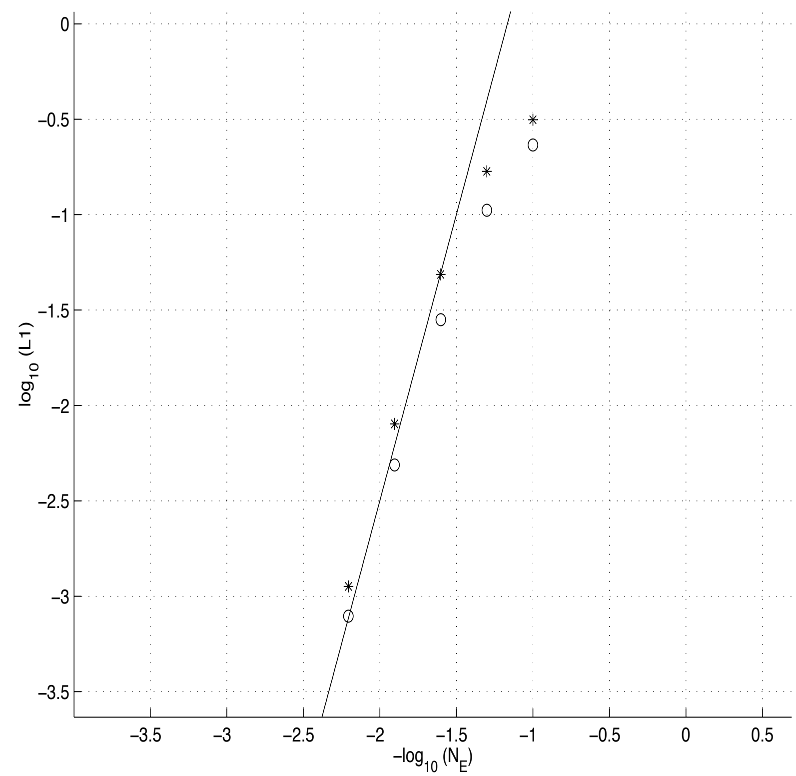
(1)Evaluate the gradient of each variable.
(2)Hermite interpolation to get mid point values.
(3)Simpson's rule to evaluate the fluctuation.

$$
\begin{aligned}
\mathbf{\Phi}_{123} &= \iint_{123} \left[\partial_x f + \partial_y g\right] dxdy = \oint_{123} f\,dy - g\,dx \\
&= \sum_{edges} \left(\overline{f}\Delta y - \overline{g}\Delta x\right) - \frac{1}{12} \sum_{edges} \left(\Delta p \Delta y - \Delta q \Delta x\right).
\end{aligned}
$$

where p and q are the directional derivatives of f and g along each edge.

A Third-Order Least-Squares

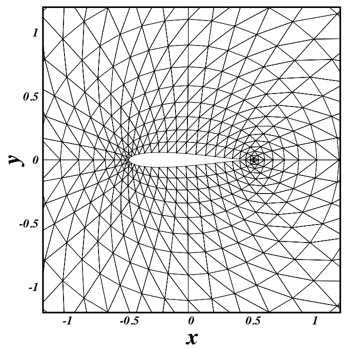




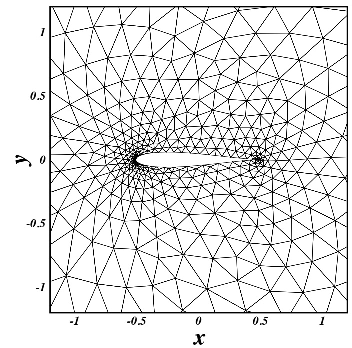
Solution obtained on an 80x40 O-grid

The convergence rate is 3

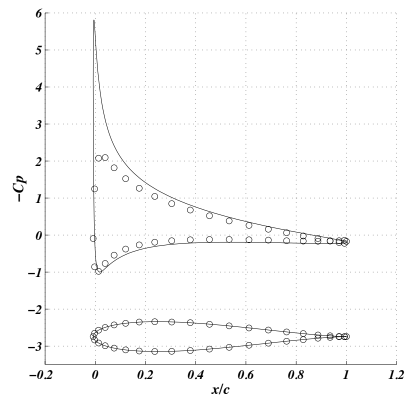
Moving Mesh Solution



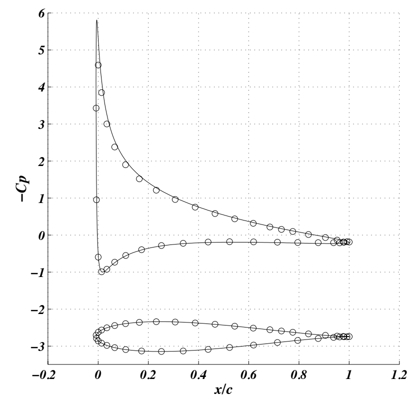
Regular 40x20 Grid



An Adaptive Grid



C_p Distribution



C_p Distribution

Conclusions and Future Work

- Residual Minimization has the Potential to be a very effective tool for grid adaptation
- Ready for the Euler Equations.
- Finding the Right Norms to Minimize.
- Node Insertion/Removal Procedure.
- Accelerate the Convergence (Multigrid).