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Third-Order Edge-Based Hyperbolic Navier-Stokes Scheme for Three-dimensional Viscous Flow

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Outline

- Introduction of third-order edge-based scheme
- Hyperbolic Navier-Stokes methods
- Source term discretization for third-order accuracy
- Numerical results
  - Accuracy verification
  - Viscous flow simulations
- Conclusion remarks and future work

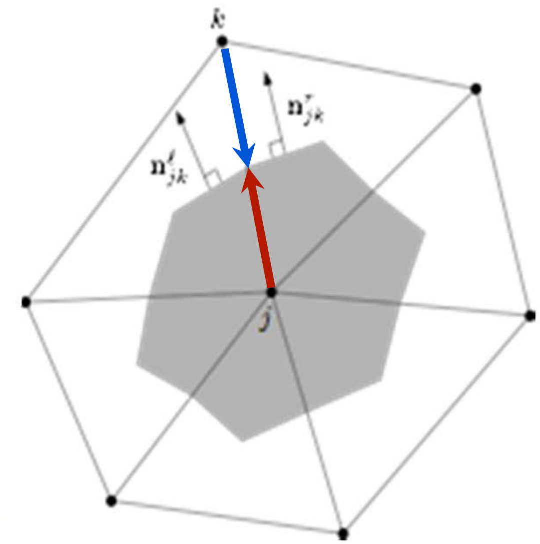
Edge-Based Discretization

- Widely used in CFD solvers like NASA's FUN3D, Software Cradle's SC/Tetra, DLR Tau code, etc.
- Achieves third-order accuracy for hyperbolic systems on arbitrary simplex-element grids with quadratic least-square (LSQ) methods and linearly-extrapolated fluxes. [1,2,3]

1. Katz and Sankaran, *J. Comput. Phys.,* Vol 230, 2011, pp. 7670-7686
2. Katz and Sankaran, *J. Sci. Comput.* Vol. 51, 2012, pp. 375-393
3. Diskin and Thomas, AIAA Paper 2012-0609

Edge-based Discretization

Arbitrary Triangular/Tetrahedral Grids



2nd-Order (Linear LSQ gradients)

$$u_L = u_j + \frac{1}{2}\overline{\nabla} u_j \cdot \Delta \mathbf{r}_{jk} \qquad \mathbf{f}_L = \mathbf{f}(u_L)$$

$$u_R = u_k - \frac{1}{2}\overline{\nabla} u_k \cdot \Delta \mathbf{r}_{jk} \qquad \mathbf{f}_R = \mathbf{f}(u_R)$$

3rd-Order (Quadratic LSQ gradients)

$$\mathbf{f}_j = \left(\frac{\partial \mathbf{f}}{\partial u}\right)_j \overline{\nabla} u_j$$

$$u_L = u_j + \frac{1}{2}\overline{\nabla} u_j \cdot \Delta \mathbf{r}_{jk} \qquad \mathbf{f}_L = \mathbf{f}_j + \frac{1}{2}\overline{\nabla}\mathbf{f}_j \cdot \Delta \mathbf{r}_{jk}$$

$$u_R = u_k - \frac{1}{2}\overline{\nabla} u_k \cdot \Delta \mathbf{r}_{jk} \qquad \mathbf{f}_R = \mathbf{f}_k - \frac{1}{2}\overline{\nabla}\mathbf{f}_k \cdot \Delta \mathbf{r}_{jk}$$

Katz&Sankaran(JCP2011)

Linear extrapolations

Economical 3rd-order scheme: edge-loop with a flux per edge

Extensions to Viscous Terms

Not straightforward due to compatibility requirement:

- Viscous terms must be discretized to guarantee[4]

$$TE(\partial_x f + \partial_y g + viscous) = \frac{h^2}{12}(\partial_x + \partial_y)^2(\partial_x f + \partial_y g + viscous) + O(h^3)$$

- *Possible, but additional complications in the algorithm…*

Or if we can write the viscous terms as a hyperbolic system:

$$\mathrm{div}\,\mathbf{f} = 0$$

then the third-order EB scheme directly applies to the viscous terms.

4. Hiroaki Nishikawa, *J. Comput. Phys.*, Vol 273, 2014, pp. 287-309

HNS: Hyperbolic Navier-Stokes[5]

$$
\begin{aligned}
\partial_\tau \rho + \mathrm{div}(\rho \mathbf{v}) &= \boxed{\mathrm{div}\, \mathbf{r},} \\
\partial_\tau (\rho \mathbf{v}) + \mathrm{div}(\rho \mathbf{v} \otimes \mathbf{v}) + \mathrm{grad} p - \mathrm{div} \boldsymbol{\tau} &= 0, \\
\partial_\tau (\rho E) + \mathrm{div}(\rho \mathbf{v} H) - \mathrm{div}(\boldsymbol{\tau} \mathbf{v}) + \mathrm{div} \mathbf{q} &= 0, \\
\frac{T_v}{\mu_v} \partial_\tau \mathbf{g} - \mathrm{grad}\, \mathbf{v} + \mathbf{g}/\mu_v &= 0, \\
\frac{T_h}{\mu_h} \partial_\tau \mathbf{q} + \mathrm{grad}\left(\frac{T}{\gamma(\gamma-1)} \right) + \mathbf{q}/\mu_h &= 0, \\
\frac{T_\rho}{\nu_\rho} \partial_\tau \mathbf{r} - \mathrm{grad} \rho + \mathbf{r}/\nu_\rho &= 0.
\end{aligned}
$$

Negligibly small

HNS Source term

Small coefficient

$$\nu_\rho = V_{min}$$

$$\nabla \rho_j = \frac{\mathbf{r}_j}{\nu_\rho}, \quad \nabla \mathbf{v}_j = \frac{\mathbf{g}_j}{(\mu_v)_j}, \quad \nabla T_j = -\gamma(\gamma-1) \frac{\mathbf{q}_j}{(\mu_h)_j}.$$

5. Y. Nakashima, N. Watanabe, H. Nishikawa, AIAA Paper 2016-1101

Advantages of Hyperbolic Method

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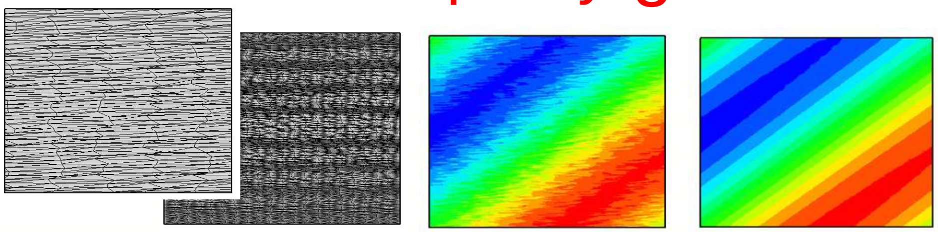
Same discretization for inviscid and viscous terms.

E.g., Edge-based, reconstruction, upwind viscous flux.

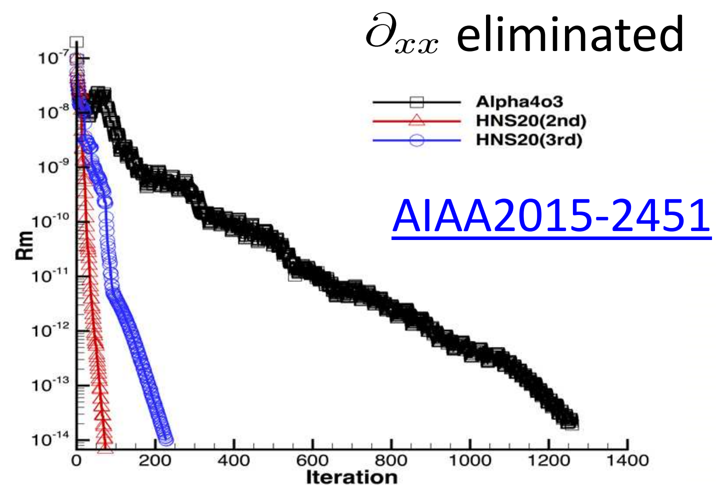
Higher-order/quality gradients



LSQ Hyperbolic

well suited for anisotropic adaptation

Higher-order inviscid scheme



See hyperbolic method website: http://hiroakinishikawa.com/fohsm/

Source Term Discretization

Compatibility Conditions with Source:

$$\frac{1}{V_j}\sum_{k\in\{k_j\}}\mathbf{\Phi}_{jk}|\mathbf{n}_{jk}| - \int_{V_j}\mathbf{S}\,dV = \mathbf{R}'_j - \frac{h^2}{12}\left[\partial_{xx}\mathbf{R}'_j + \partial_{yy}\mathbf{R}'_j + \partial_{zz}\mathbf{R}'_j - \partial_{xy}\mathbf{R}'_j - \partial_{yz}\mathbf{R}'_j + \partial_{zx}\mathbf{R}'_j\right] + O(h^3)$$

$$\mathbf{R}' = \mathrm{div}\mathbf{F} - \mathbf{S}$$

$$\frac{1}{V_j}\sum_{k\in\{k_j\}}\mathbf{\Phi}_{jk}|\mathbf{n}_{jk}| - \int_{V_j}\mathbf{S}\,dV = \mathrm{div}\mathbf{F} - \mathbf{S} + O(h^3)$$

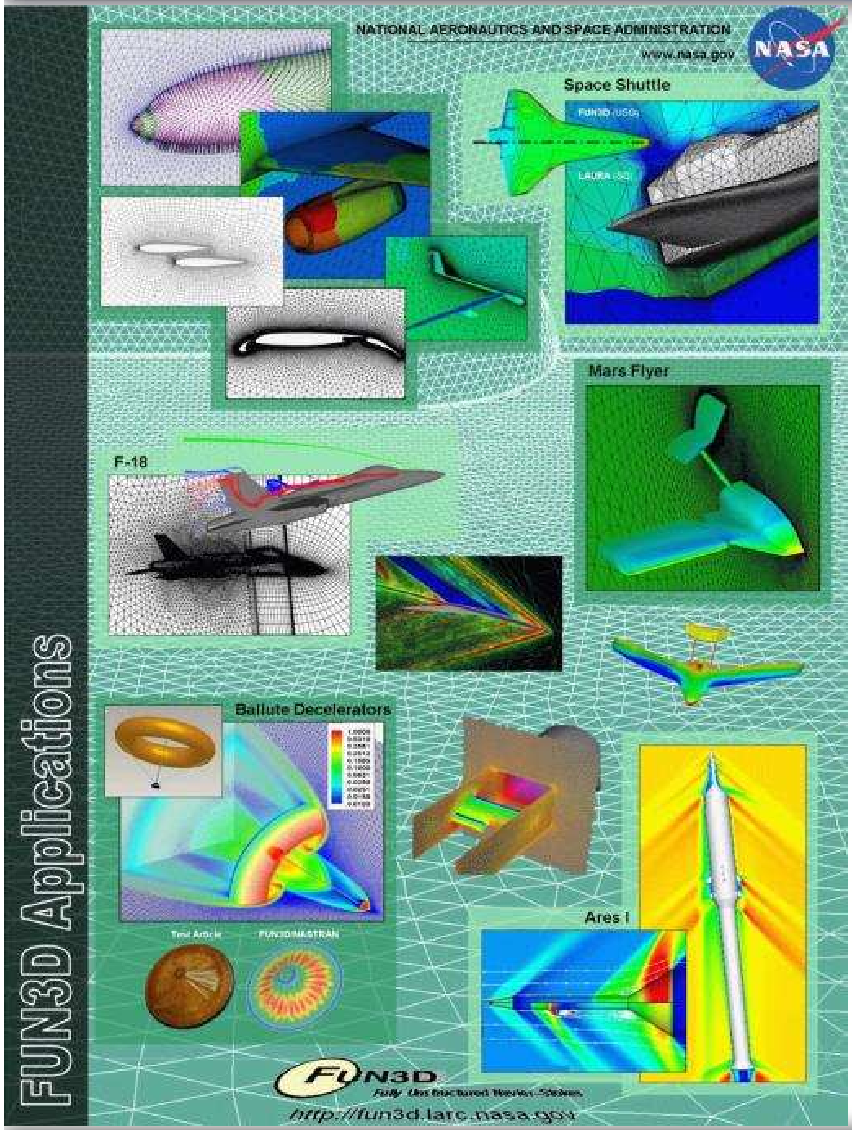
Various source term quadrature formulation exist that satisfy this property. In this work, the most economical formula is used.[6]

$$\int_{V_j}\mathbf{S}\,dV = \sum_{k\in\{k_j\}}\frac{(13\mathbf{S}_j + 3\,\partial_{jk}\mathbf{S}_j - 3\mathbf{S}_k)\,(\Delta\mathbf{x}_{jk}\cdot\mathbf{n}_{jk})}{60}$$

$$\nabla\mathbf{S}_j = [0,\,0,\,0,\,0,\,0,\,\nabla(\mathbf{g}_u/\mu_v),\,\nabla(\mathbf{g}_v/\mu_v),\,\nabla(\mathbf{g}_w/\mu_v),\,\nabla(\mathbf{q}/\mu_h),\,\nabla(\mathbf{r}/\nu_\rho)]_j^t$$

6. H. Nishikawa, Y. Liu, , *J. Comput. Phys.*, Vol 344, 2017, p595-622

Third-order Edge-based Schemes in FUN3D



- FUN3D, developed by NASA LaRC, is chosen as the CFD solver for the implementation of the third-order scheme.
- FUN3D-i3rd, the third-order-inviscid scheme plus the Galerkin discretization for viscous terms.
- FUN3D-HNS20 (HNS20), the third-order scheme with Hyperbolic Navier-stokes viscous discretization.

Third-order Edge-based Schemes in FUN3D

- Implemented 3rd-order inviscid scheme into FUN3D with linear inviscid flux (roe-flux) extrapolation, 2nd-order accurate gradients (from QLSQ or HNS gradient variables) and 3rd-order boundary flux quadrature. [7]
- Extended 3rd-order inviscid scheme into unsteady simulation with applying accuracy preserving source term quadrature for the source terms from time derivatives. [8]
- Current effort is to extend 3rd-order edge-based scheme to full Navier-Stokes equations with applying HNS method including linear HNS viscous flux extrapolation, QLSQ for all HNS variables, and source term quadrature for HNS source terms.

7. Y. Liu, H. Nishikawa, AIAA Paper 2016-3969
8. Y. Liu, H. Nishikawa, AIAA Paper 2017-0738

Summary of Discretization

Scheme	Discretization: Inviscid Flux		Inviscid LSQ $(\rho, \mathbf{v}, p)$	Viscous Flux		Viscous LSQ $(\mathbf{r}, \mathbf{g}, \mathbf{q})$	HNS Source
FUN3D	Roe(2nd)	: 2	Linear	Galerkin(2nd)	: 1	None	None
FUN3D-i3rd	Roe(3rd)	: 3	Quadratic	Galerkin(2nd)	: 1	None	None
HNS20-IQ(2nd)	Roe(3rd)	: 2	C-quadratic	Upwind(2nd)	: 2	Linear	Point
HNS20-II(2nd)	Roe(3rd)	: 1	N/A	Upwind(2nd)	: 2	Linear	Point
HNS20-IQ(3rd)	Roe(3rd)	: 3	C-quadratic	Upwind(3rd)	: 3	Quadratic	Compact
HNS20-II(3rd)	Roe(3rd)	: 1	N/A	Upwind(3rd)	: 3	Quadratic	Compact

HNS20-II(3rd) : Use $(\mathbf{r}, \mathbf{g}, \mathbf{q})$ - 2nd-order accurate

HNS20-IQ(3rd) : Compact Quadratic LSQ gradients[6]

Numerical Results

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- Accuracy verification with manufactured solutions
- Viscous laminar flow simulations
  - Low Reynolds number flow over a sphere
  - Middle Reynolds number flow over a Joukowsky airfoil (in paper)
  - High Reynolds number flow over a flat plate

Accuracy Verification, Re = 1

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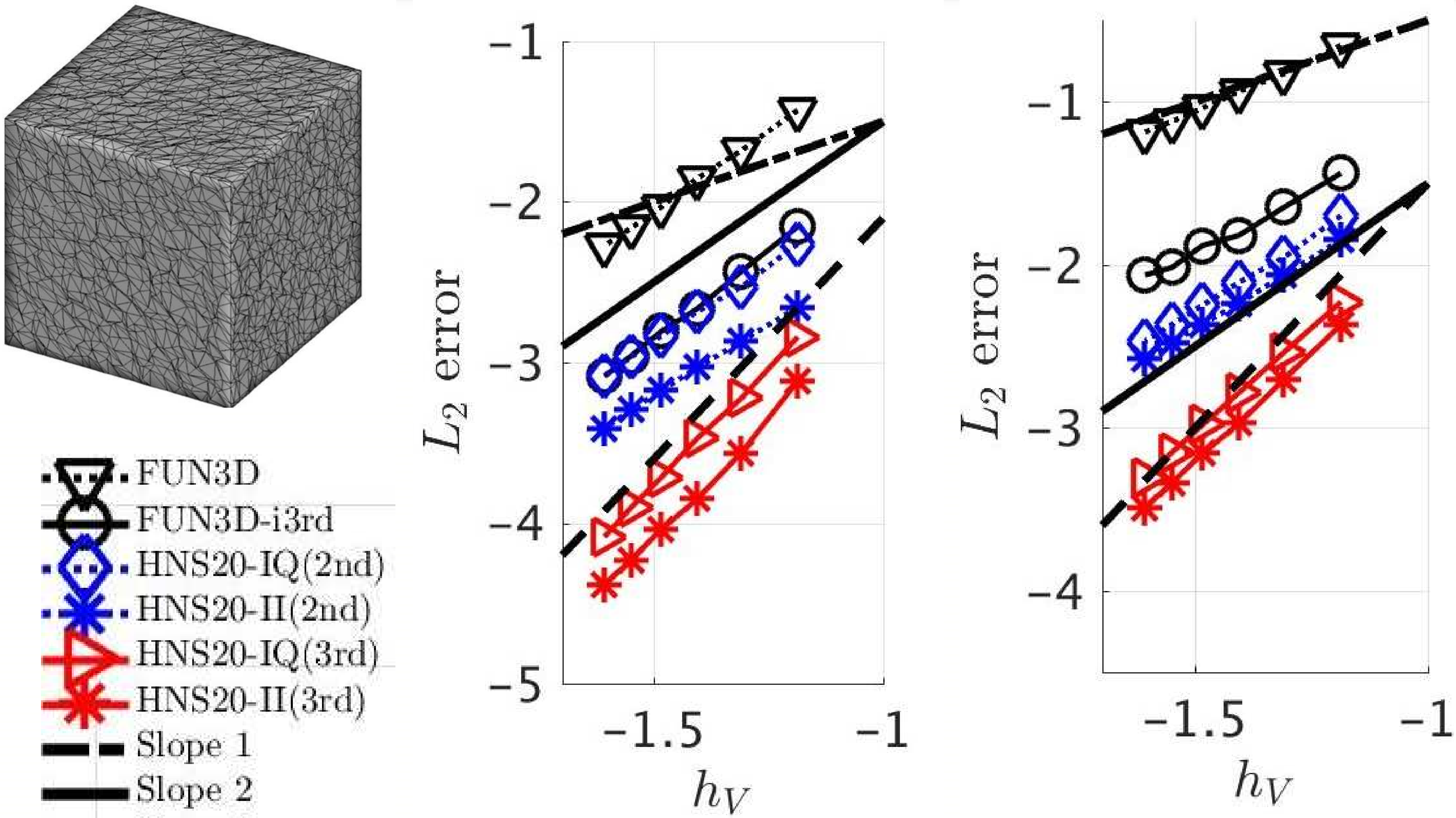
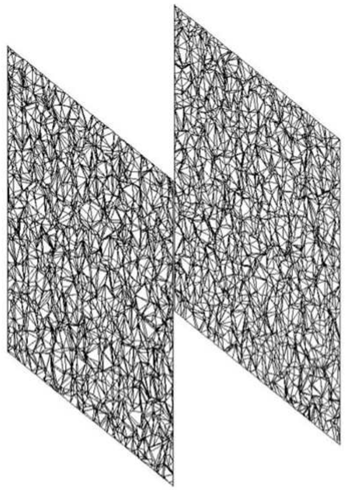
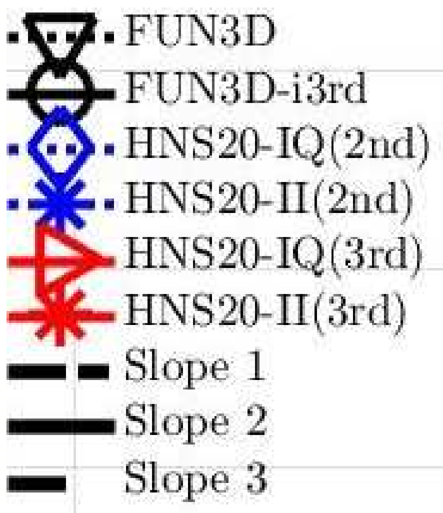


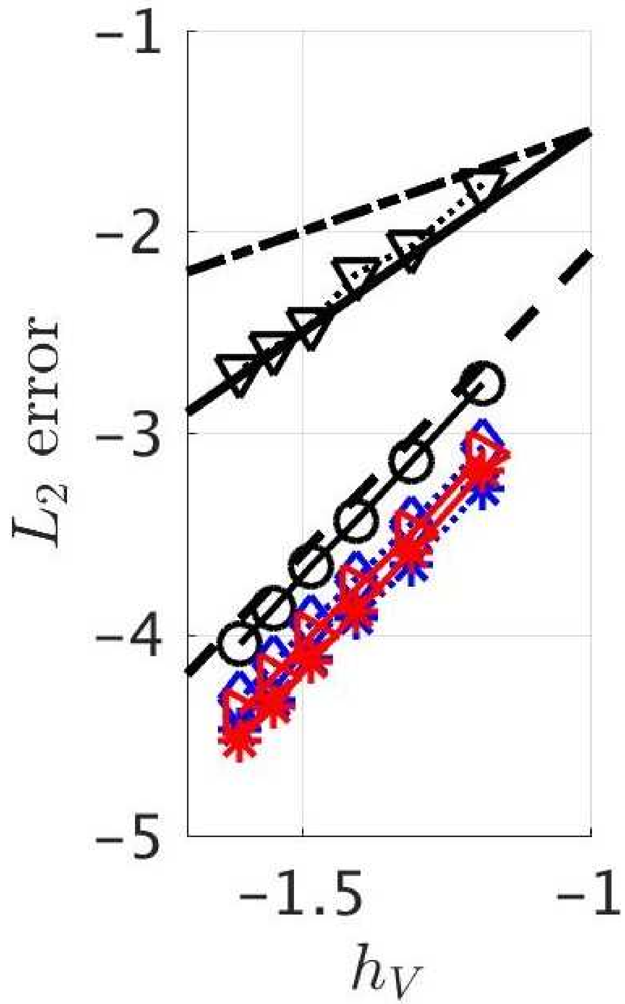
Pressure

Pressure gradient, $\partial_x p$

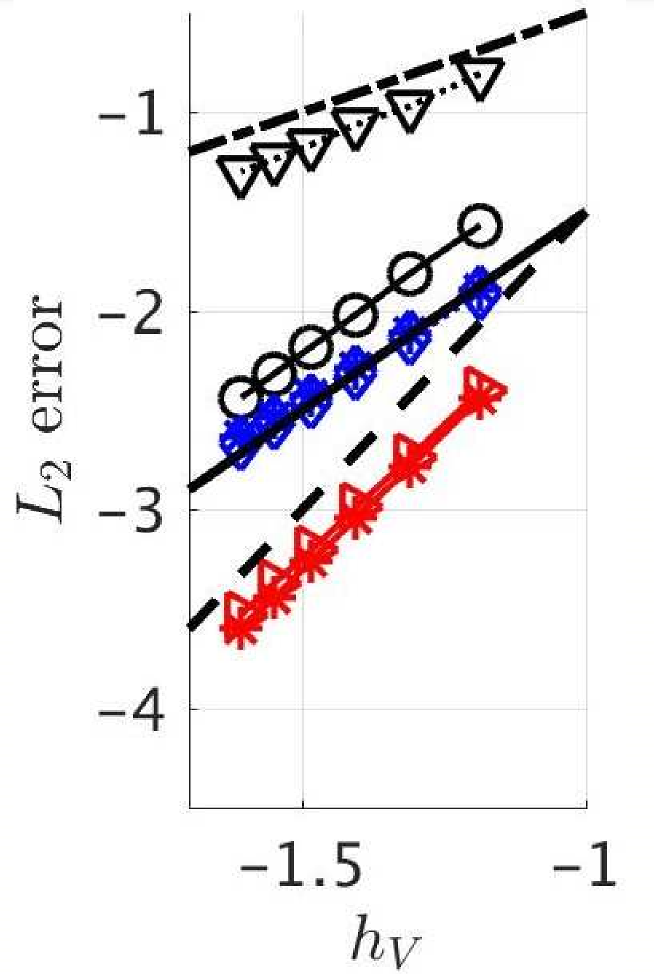
Accuracy Verification, Re = 10000







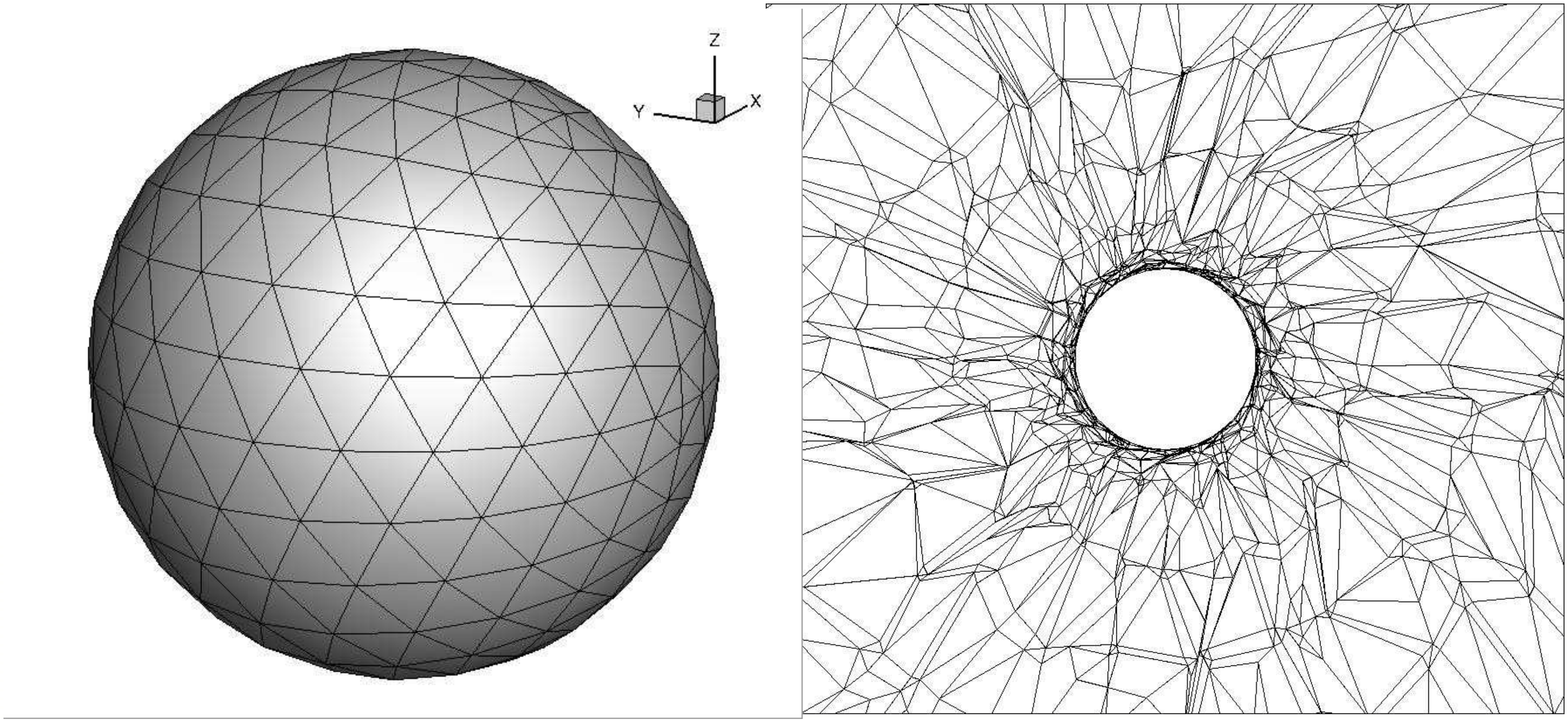
Pressure



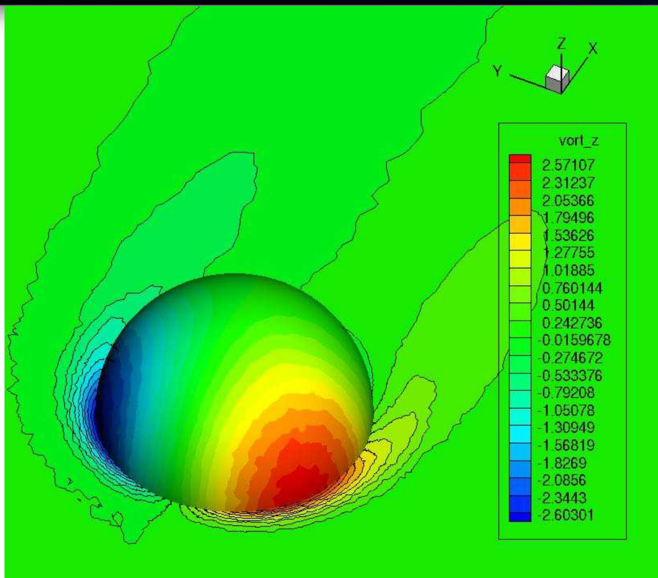
Pressure gradient, $\partial_x p$

Sphere at Re=101, Mach = 0.15

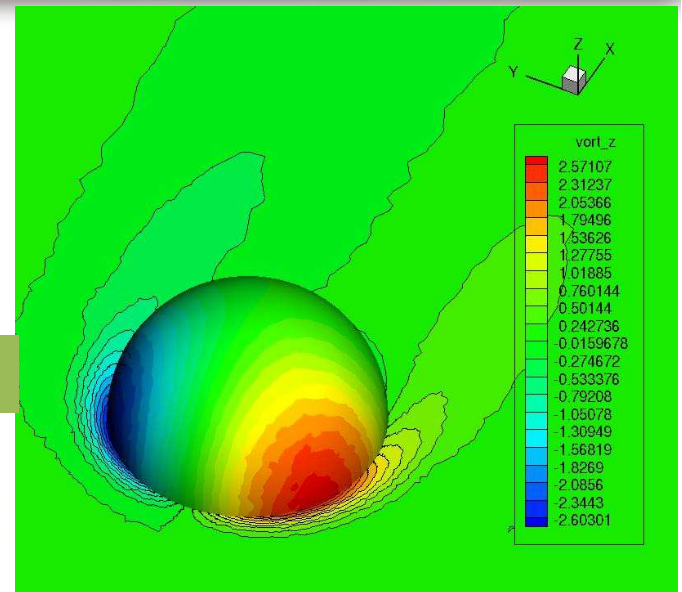
4 tetrahedral grids: 15.7k – 1.16mil nodes



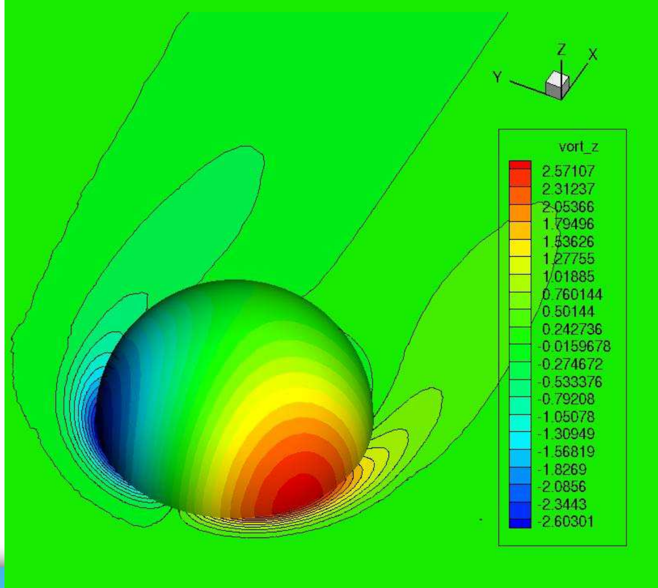
Vorticity(z) Contour (finest grid)



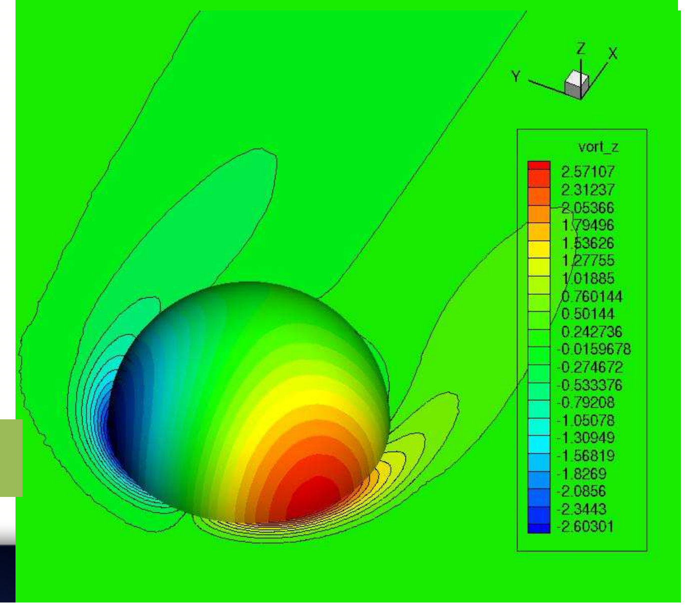
FUN3D



FUN3D-i3rd

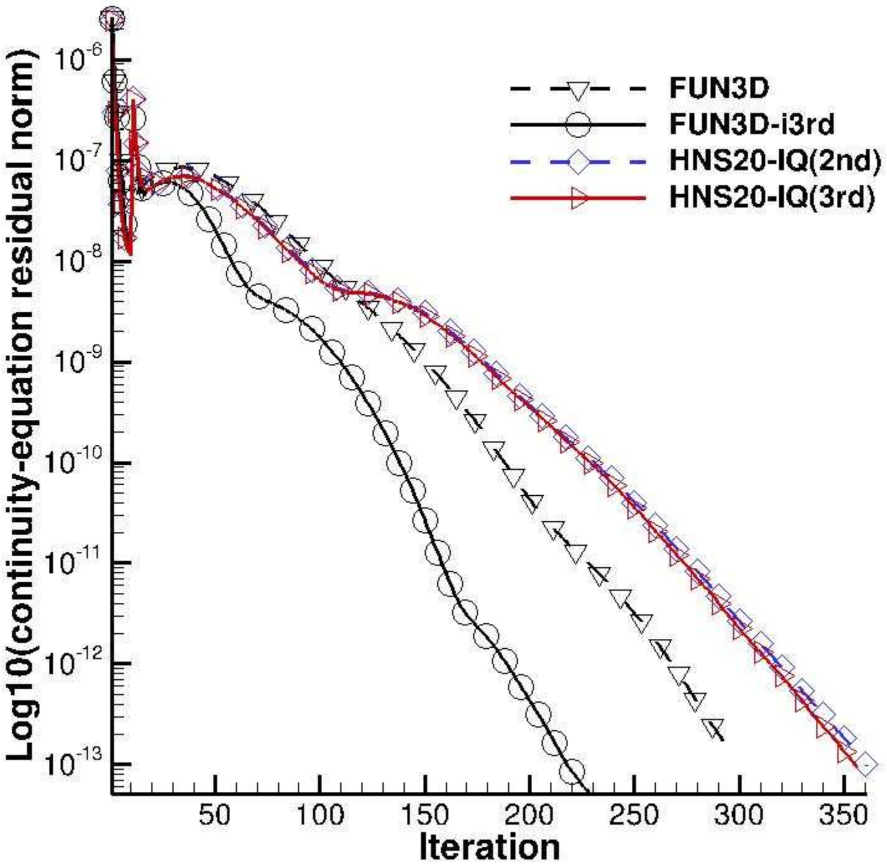


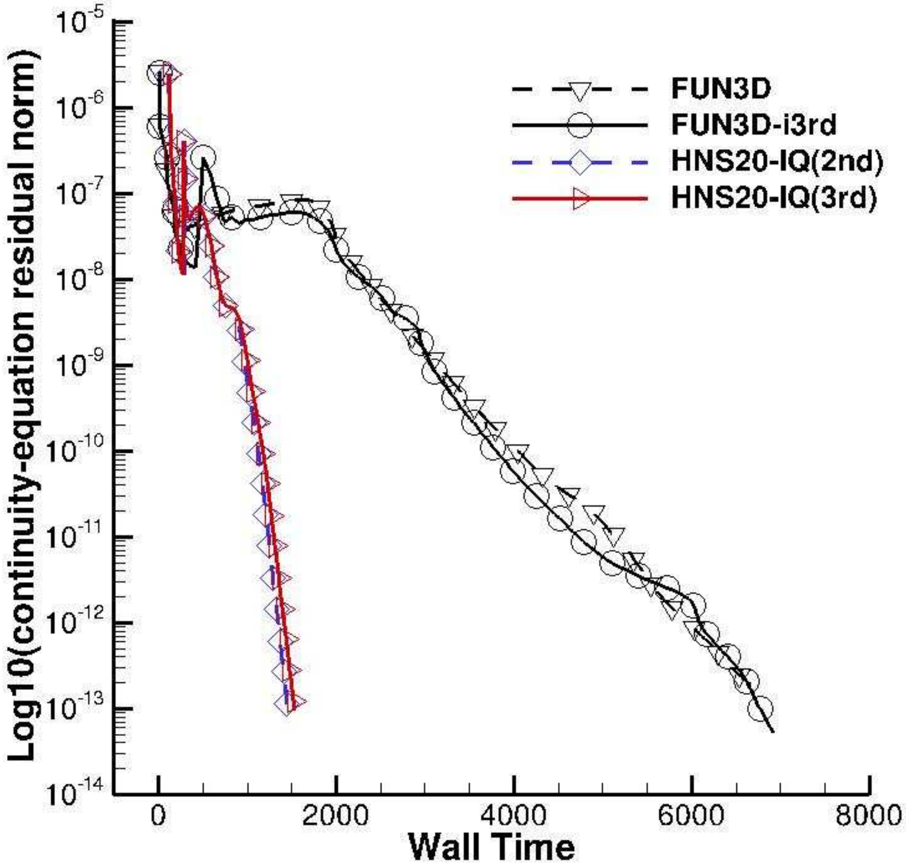
HNS20-IQ(2nd)



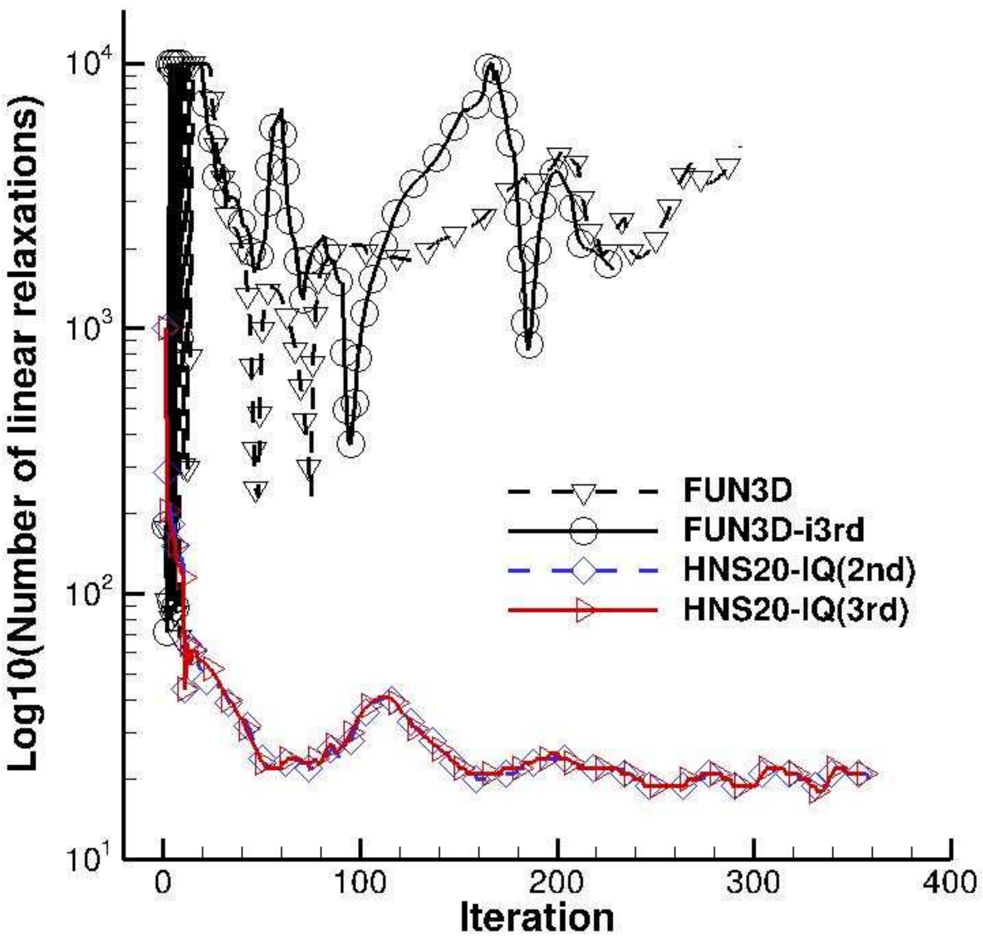
HNS20-IQ(3rd)

Convergence results for finest grid





Number of linear relaxation



- The number of linear relaxation is determined by reduction of linear residual by one order of magnitude
- FUN3D and FUN3D-i3rd will not converge to machine zero without linear relaxation error control.

Flat plate at Rey = 1 million

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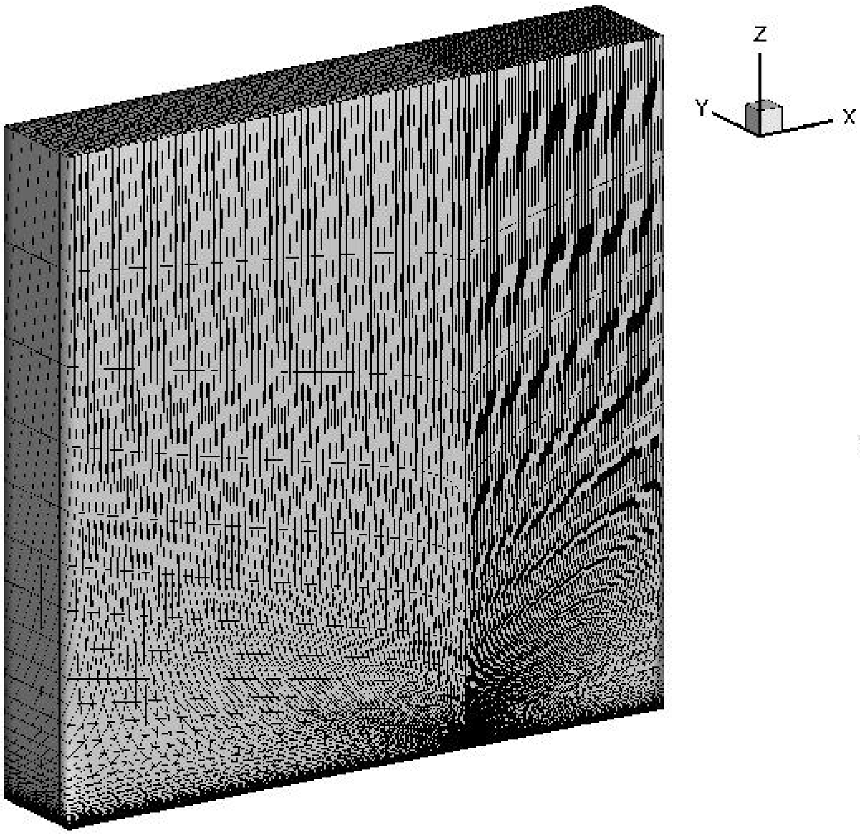
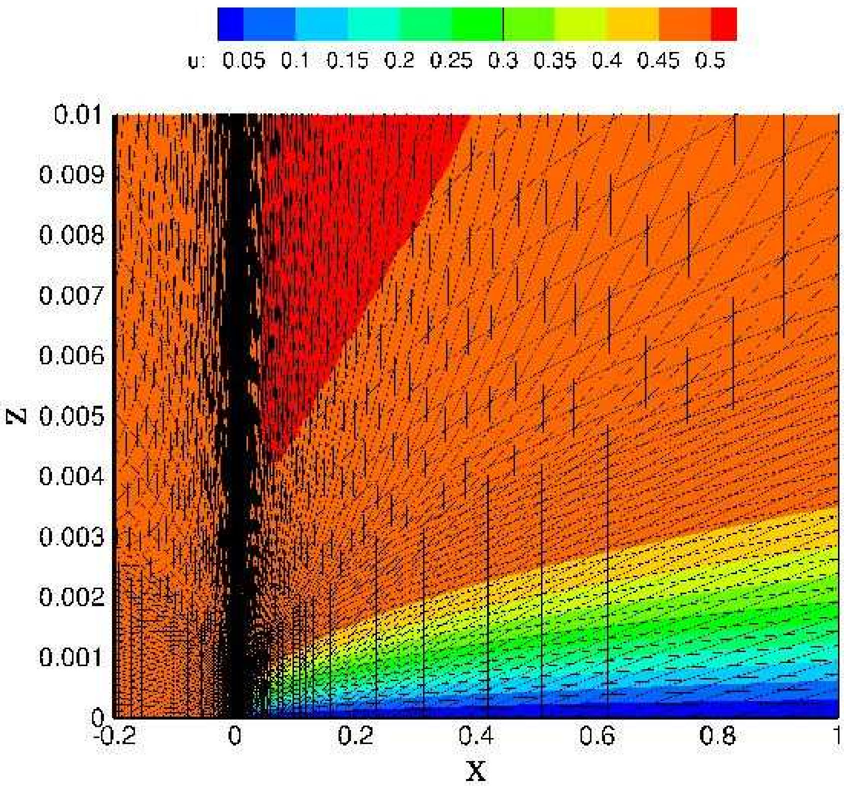
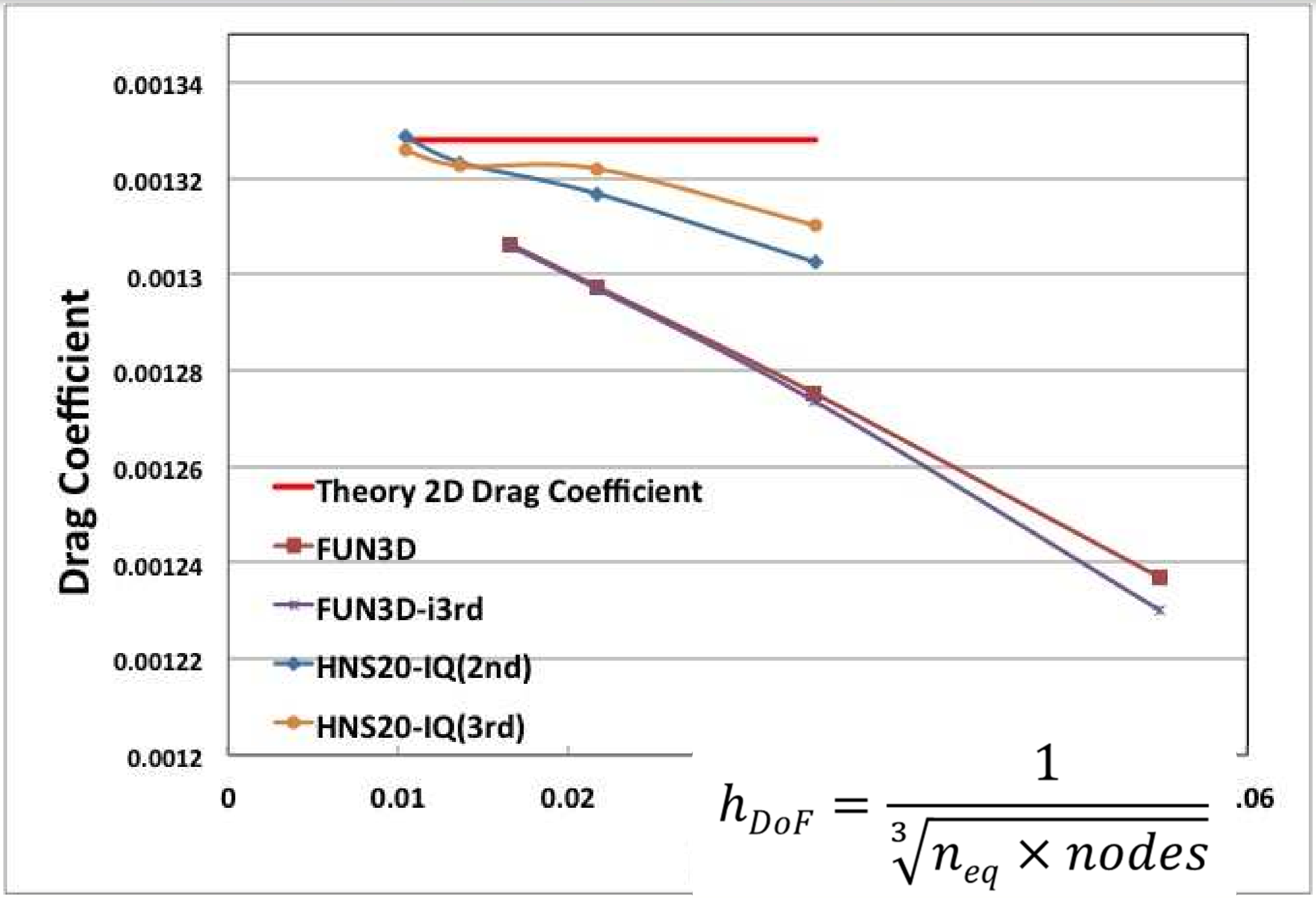




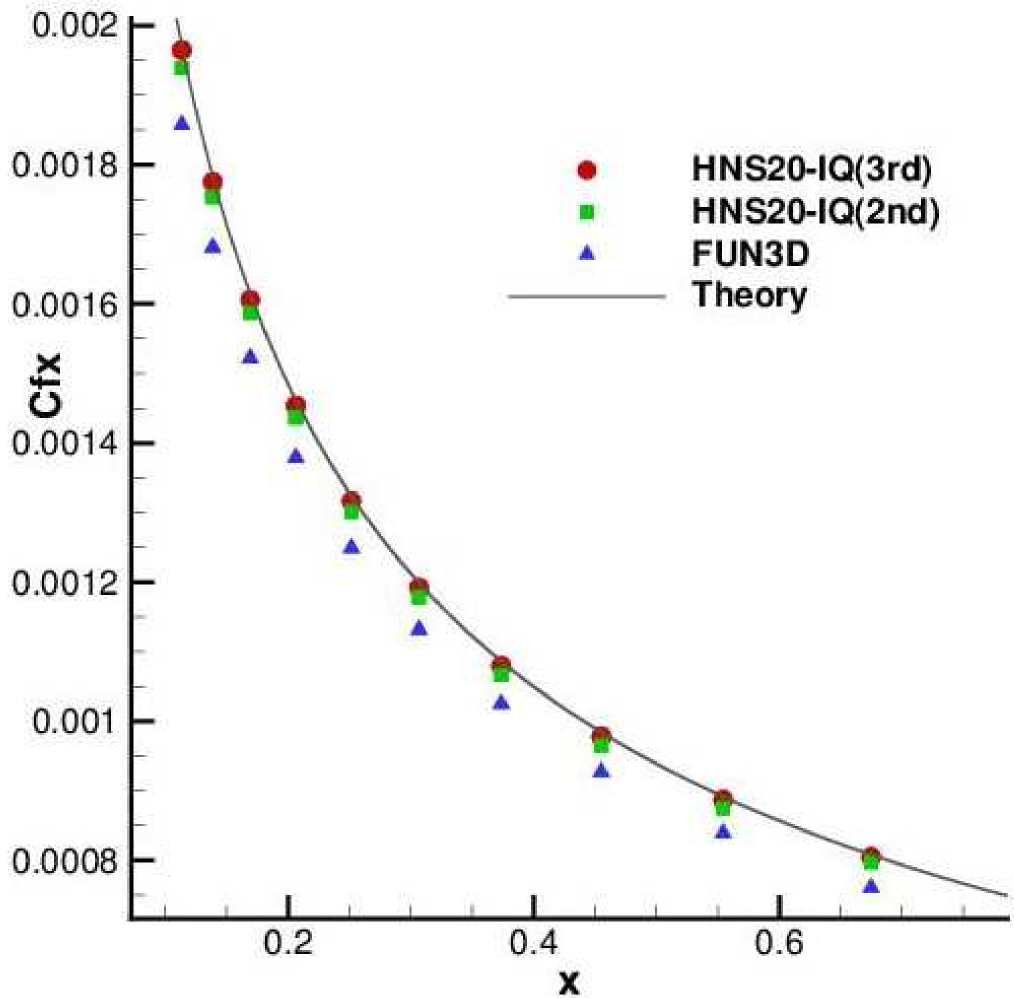
4 tetrahedral grids: 64x32, 128x64, 256x128, 384x192

Drag coefficient grid convergence



$$h_{DoF} = \frac{1}{\sqrt[3]{n_{eq} \times nodes}}$$

Skin friction at coarsest grid



Conclusions

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- Extended third-order edge based scheme to the Navier-Stokes equations for tetrahedral grids by using a hyperbolic viscous formulation.
- Source terms from the hyperbolic viscous formulation has been discretized by an efficient accuracy-preserving quadrature formula, which does not require second derivatives of the source terms.
- Due to the hyperbolic formulation, the resulting third-order Navier-Stokes scheme shown to achieve 3rd order accuracy for the primitive variables and their gradients.

Future Work

- Extend HNS20 to turbulent flow

- Develop a more smart quadratic LSQ fit method

- Make the HNS20 method more efficient in Navier-stokes computations

Acknowledgments

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