# High-Order Residual-Distribution Schemes for Discontinuous Problems on Irregular Triangular Grids 

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#### Abstract

In this paper, we construct second- and third-order non-oscillatory shock-capturing hyperbolic residual-distribution schemes for irregular triangular grids, extending the schemes developed in J. Comput. Phys., 300 (2015), 455-491 to discontinuous problems. We present extended first-order N - and Rusanov-scheme formulations for a hyperbolic advectiondiffusion system, and demonstrate that the hyperbolic diffusion term does not have any adverse effect on the solution of inviscid problems for a vanishingly small viscous coefficient. We then construct second- and third-order non-oscillatory hyperbolic residualdistribution schemes by blending the non-monotone second- and third-order schemes with the extended first-order schemes as typically done in the residual-distribution schemes, and examine them for discontinuous problems on irregular triangular grids. We also propose to use the Rusanov scheme to avoid non-physical shocks in combination with an improved characteristics-based nonlinear wave sensor for detecting shocks, compression, and expansion regions. We then verify the design order of accuracy of these blended schemes on irregular triangular grids.


## I. Introduction

Accurate detection of discontinuities are of great interest to many practical applications. Equally, accurate prediction of solution and solution gradients in the smooth regions on irregular grids are also essential in estimating many important physical quantities such as viscous stresses, vorticity, and heat flux. In Ref. 1, we presented new second- and third-order hyperbolic advection-diffusion residual-distribution (RD) schemes called the RD-CC2 and RD-CC3 schemes, respectively, and demonstrated that these schemes predict solution and solution gradients efficiently (typically 10-15 residual evaluations to obtain a converged solution) and accurately on anisotropic and irregular triangular grids. These schemes are constructed based on the hyperbolic method, ${ }^{2}$ where the diffusion term is formulated as a hyperbolic system by including the solution gradients as extra variables, but with a new design principle that ensures the cell residual vanishes for exact quadratic (for second-order) and cubic (for third-order) solutions for arbitrary triangular elements. The new design principle was proposed in Ref. 1 and found critical for smooth and accurate predictions of solution gradients both on the physical geometry and within the domain for highly irregular elements. These schemes also produce solution and solution gradients with an equal order of accuracy on fully irregular elements. These schemes are, however, produce oscillatory solution around discontinuities (such as a shock) and therefore, some special treatment is needed to prevent such oscillatory solutions. The objective of the work presented in this paper is, therefore, to develop non-oscillatory versions of the RD-CC2 and RD-CC3 schemes with a mechanism to avoid entropy-violating shocks.

Before discussing strategies for constructing non-oscillatory schemes, we remark that the RD-CC2 and RD-CC3 schemes have an unconventional feature compared with other RD schemes. These schemes have the advective term coupled with the diffusive term even in the advection limit via the extra variables introduced to formulate diffusion as a hyperbolic system. In other words, vanishingly small diffusion coefficient results

[^0]in a pure advection scheme coupled with the equations for the extra variables. The coupling is advantageous becasue it helps improve the order of accuracy of the advection scheme, typically, by one order. ${ }^{1}$ However, a conventional non-oscillatory technique developed for the advection equation, which does not take into account the coupling, may not be immediately applicable to the RD-CC2 and RD-CC3 schemes. For these reasons, we consider the advection-diffusion equation throughout the paper although our target problems are purely advective, and seek a non-oscillatory technique that is simple and easy to apply to these schemes.

For the construction of non-oscillatory schemes, few approaches have been proposed that are widely used within the RD community. These are: (1) nonlinear advection schemes such as the modified N scheme or the Positive-Streamwise-Invariant (PSI) scheme, ${ }^{3}$ and limited schemes, ${ }^{4}$ where a high-order smooth solution is recovered from a first-order positive scheme with a smoothness indicator, and (2) blended schemes, ${ }^{5}$ in which a first-order and high-order schemes are blended through a nonlinear blending function. Although these approaches are different, one may recover an identical scheme from either of these approaches. ${ }^{6,7}$ These nonlinear schemes are first developed for the scalar advection equation and later extended for a system of equations. ${ }^{8,9}$ These schemes are widely used within the RD community, ${ }^{6,7,10-12}$ and applied to advection and advection-diffusion, ${ }^{13-16}$ steady inviscid, ${ }^{12,17,18}$ steady Navier-Stokes, ${ }^{19}$ turbulent compressible flows, ${ }^{20}$ and unsteady ${ }^{12,21,22}$ problems. It may also be possible to employ an artificial viscosity technique as widely used in the stabilized finite-element methods, e.g., Refs., ${ }^{23,24}$ because the RD schemes can be formulated as Petrov-Galerkin schemes.

Among various approaches, in this paper, we consider the blending approach ${ }^{5,7,12}$ for constructing nonoscillatory RD-CC2 and RD-CC3 schemes. Specifically, we construct a monotone first-order scheme for the hyperbolic advection-diffusion system by applying first-order RD schemes known to be monotone for hyperbolic systems, the N -scheme and the Rusanov scheme, and then blend it with the RD-CC2 and RDCC3 schemes through a nonlinear blending function similar to the one presented in, e.g., Ref. 5. This strategy has been found simple, systematic, and also convenient as entropy-violating shocks can be avoided within the same framework. Other approaches may also be explored, but the comparison of different approaches is beyond the scope of the paper and thus left as future work.

We also propose a technique to avoid entropy-violating shocks by activating the first-order Rusanov scheme at sonic expansion. This approach requires accurate detection of sonic expansion. We therefore, perform this task by developing a new characteristics-based nonlinear wave sensor to accurately detect sonic expansion. The presented technique is an improvement to the technique reported in Refs. 25, 26, which uses divergence of the steady characteristics as a mechanism to identify whether an element is in a shock, rarefaction, or away from such nonlinear waves. The technique of Refs. 25,26 , however, requires a threshold, and that is often very difficult to know a priori; a large threshold causes instability by high-order methods, while small thresholds lead compression waves to be treated as shocks, which in turn make the solution less accurate and undesirable. Here, we improve this technique with a more accurate characteristics-based nonlinear wave operator that is less dependent on such thresholds. In this present work, the proposed characteristicbased sensor is used as an alternative approach to a more traditional entropy fix technique ${ }^{7,27}$ for avoiding unphysical shocks (entropy-violating solutions). Another alternative is to use a special quadrature formula, ${ }^{26}$ but this technique requires development of completely new high-order schemes and therefore, is not pursued in this study. The proposed sensor may also be used as a first step toward the development of a shock-fitting scheme. ${ }^{28-30}$

In this paper, we focus on two-dimensional hyperbolic advection-diffusion systems and develop secondand third-order blended hyperbolic residual-distribution schemes for discontinuous problem on irregular triangular grids. We first demonstrate that the hyperbolic diffusion terms do not negatively affect the solution of the advection equation as the diffusion coefficient approaches zero. We then demonstrate that these blended schemes can successfully detect physical discontinuities using the developed characteristicsbased nonlinear wave sensor, and avoid unphysical shocks when the proposed extended Rusanov scheme is used as a first-order advection-diffusion scheme. Through numerical examples, we show that the proposed schemes not only provide an accurate solution but also give accurate and smooth solution gradients (away from discontinuities) on irregular grids. This is extremely important because, as we will demonstrate, least squares reconstruction of gradients, which is commonly used, could be very inaccurate and oscillatory even if a high-order solution is used as a basis for the gradient reconstruction.

The paper is organized as follows. In Section II, we briefly describe the basics of a nonlinear hyperbolic advection-diffusion system. In Section III, we present extended first-order N and Rusanov schemes for a general hyperbolic advection-diffusion system. In Section IV, we review the baseline hyperbolic RD, and
the second- and third-order hyperbolic RD schemes, proposed in Ref. 1; we use these schemes for the construction of high-order blended schemes, which are presented in Section V. In Section VI, we discuss how entropy-violating solutions can be avoided, followed by a boundary condition formulation in Section VII. We then present numerical examples in Section VIII, demonstrating the shock-capturing capability of the constructed second- and third-order blended hyperbolic advection-diffusion RD schemes on irregular triangular grids. Order of accuracy of these blended schemes are also verified in this section. We then summarize the presented work with some concluding remarks in Section IX.

## II. General nonlinear hyperbolic advection-diffusion system and discretization

Consider the following general two-dimensional nonlinear advection-diffusion equation:

$$
\begin{equation*}
\partial_{t} u+\partial_{x} f+\partial_{y} g=\partial_{x}\left(\nu \partial_{x} u\right)+\partial_{y}\left(\nu \partial_{y} u\right)+\tilde{s}(x, y, u) \tag{1}
\end{equation*}
$$

where $f$ and $g$ are nonlinear functions of $u, \nu=\nu(u)$, and $\tilde{s}(x, y, u)$ denotes a source term. In this work, we only consider a constant diffusion coefficient, but the schemes are presented in the form directly applicable to nonlinear diffusion coefficient. The advection speeds in $x$ and $y$ directions are therefore $a(u)=\partial f / \partial u$ and $b(u)=\partial g / \partial u$, respectively. We reformulate the advection-diffusion equation in the form of a nonlinear hyperbolic advection-diffusion system using a preconditioning matrix $\mathbf{P}$, which is to simplify the construction of the numerical scheme: ${ }^{31}$

$$
\begin{equation*}
\mathbf{P}^{-1} \frac{\partial \mathbf{U}}{\partial \tau}+\frac{\partial \mathbf{F}}{\partial x}+\frac{\partial \mathbf{G}}{\partial y}=\mathbf{Q} \tag{2}
\end{equation*}
$$

where $\mathbf{U}=[u, p, q]^{T}$, where the superscript $T$ indicates the transpose, and

$$
\begin{gather*}
\mathbf{P}^{-1}=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & T_{r} / \nu(u) & 0 \\
0 & 0 & T_{r} / \nu(u)
\end{array}\right], \quad \mathbf{F}=\mathbf{F}^{a}+\mathbf{F}^{d}=\left[\begin{array}{l}
f \\
0 \\
0
\end{array}\right]+\left[\begin{array}{c}
-p \\
-u \\
0
\end{array}\right],  \tag{3}\\
\mathbf{Q}=\mathbf{Q}^{s}+\mathbf{Q}^{d}=\left[\begin{array}{c}
s(x, y, u) \\
0 \\
0
\end{array}\right]+\left[\begin{array}{c}
0 \\
-p / \nu(u) \\
-q / \nu(u)
\end{array}\right], \quad \mathbf{G}=\mathbf{G}^{a}+\mathbf{G}^{d}=\left[\begin{array}{c}
g \\
0 \\
0
\end{array}\right]+\left[\begin{array}{c}
-q \\
0 \\
-u
\end{array}\right], \tag{4}
\end{gather*}
$$

where $\tau$ is the pseudo-time, and $T_{r}=L^{2} / \nu$ is the relaxation time with length scale defined as $L=1 / 2 \pi$. In the pseudo-steady state, the variables $p$ and $q$ corresponds to the diffusive fluxes or equivalently the solution gradients scaled by the diffusion coefficient in $x$ and $y$ directions, respectively. Thus, the above system becomes identical to the original governing equation with an implicit time-stepping technique ${ }^{32}$ for discretization of the physical time derivative, which is added to $\tilde{s}(x, y, u)$ to define $s(x, y, u)$ (see Refs. 32,33 for more details). We remark that this approach for treating time dependent term is only one of a possibility, and other techniques may also be employed (see e.g., ${ }^{34}$ ).

We divide the computational domain into a set $\{E\}$ of arbitrary triangular elements, $E$, and an associated set $\{J\}$ of nodes (or vertices). We store the solution and solution gradients, $\left(u_{j}, p_{j}, q_{j}\right)$, at each global node $j \in\{J\}$, and discretize the hyperbolic advection-diffusion system by the RD method.

The wave structure of the above hyperbolic advection-diffusion system is characterized by the eigenstructure of the preconditioned flux Jacobian $\mathbf{A}_{n_{j}}$ expressed as:
where the over-bar denotes a value evaluated by the arithmetic average of the solution $\mathbf{U}$ over the three nodes in the triangular element, $\mathbf{n}_{j}=\left(n_{x_{j}}, n_{y_{j}}\right)$ and $\hat{\mathbf{n}}_{j}=\left(\hat{n}_{x_{j}}, \hat{n}_{y_{j}}\right)$ are, respectively, the scaled and unit inward normal vectors of the edge opposite to the node $j$ within the element $E$. Here, the preconditioned flux Jacobian is split to the preconditioned advection and diffusion fluxes Jacobian; i.e., $\mathbf{A}_{n}^{a}$ and $\mathbf{A}_{n}^{d}$. The advection wave speed is $\lambda^{a}=a \hat{n}_{x}+b \hat{n}_{y}$, while the diffusion wave speeds are $\lambda_{1}^{d}=-\sqrt{\bar{\nu} / T_{r}}$ and $\lambda_{2}^{d}=\sqrt{\bar{\nu} / T_{r}}$.

Briefly, in the RD method, the preconditioned cell residual, $\boldsymbol{\Phi}^{E}$, is first computed over a triangular
element $E$ as an integral approximation of the target equations, ${ }^{1}$

$$
\boldsymbol{\Phi}^{E}=\left[\begin{array}{c}
\Phi_{u}^{E}  \tag{6}\\
\Phi_{p}^{E} \\
\Phi_{q}^{E}
\end{array}\right]=\overline{\mathbf{P}}\left(-\frac{1}{2} \sum_{j \in E}\left(\mathbf{F}_{j} \hat{n}_{x_{j}}+\mathbf{G}_{j} \hat{n}_{y_{j}}\right)\left|\mathbf{n}_{j}\right|+\frac{1}{3} \sum_{j \in E} \mathbf{Q}_{j} d \Omega^{E}\right)
$$

where $d \Omega^{E}$ is the area of the triangle $E$.
The cell residual $\boldsymbol{\Phi}^{E}$ is then split and distributed to the element vertices using a local multi-dimensional upwinding distribution function, $\mathbf{B}_{j}^{E}$, as

$$
\begin{equation*}
\boldsymbol{\Phi}_{j}^{E}=\mathbf{B}_{j}^{E} \boldsymbol{\Phi}^{E} \tag{7}
\end{equation*}
$$

where $\boldsymbol{\Phi}_{j}^{E}$ is the split cell-residual contributing to the node $j$ from the element $E$, and $\sum_{j \in E} \mathbf{B}_{j}^{E}=\mathbf{I}$, where $\mathbf{I}$ is the identity matrix, which ensures conservation. In general, the RD scheme is characterized by the nodal contribution from each element, and therefore, without any ambiguity and loss of generality, we drop the superscript $E$ from $\boldsymbol{\Phi}_{j}^{E}$, and refer to $\boldsymbol{\Phi}_{j}$ as the nodal contribution from each element; we use the superscript to indicate the name of the scheme.

At the end of the process, we obtain the nodal residual as

$$
\begin{equation*}
\boldsymbol{\operatorname { R e s }}_{j}=\frac{1}{d \Omega_{j}} \sum_{E \in\left\{E_{j}\right\}} \boldsymbol{\Phi}_{j} \tag{8}
\end{equation*}
$$

where $\left\{E_{j}\right\}$ denotes the set of triangles that share the node $j$, and $d \Omega_{j}$ is the median dual volume (area in 2 D ) around the node $j$. The resulting global system of the nodal-residual equations is solved by an implicit solver with the residual Jacobian constructed based on a compact second-order scheme as described, in detail, in Ref. 1.

In the next section, we describe extended first-order hyperbolic advection-diffusion schemes, which are used in constructing blended hyperbolic advection-diffusion schemes in Sec. V.

## III. First-order advection-diffusion schemes

In this section, we present extensions of the first-order N and Rusanov scheme to advection-diffusion problems. We use these extended first-order schemes to construct second- and third-order blended hyperbolic advection-diffusion schemes in Sec. V. These first-order schemes are also used in Sec. VIII to study the effects of the hyperbolic diffusion term in the solution of discontinuous advection-diffusion problems as the diffusion coefficient approaches zero.

## A. Advection-diffusion $N$ scheme

In this section, we directly extend the original first-order advection $N$ (aka Narrow) scheme proposed in Refs. 35,36 for inhomogeneous advection equations to a general hyperbolic advection-diffusion system.

Consider an advection system: $\partial_{t} \mathbf{U}+\partial_{x} \mathbf{F}^{a}+\partial_{y} \mathbf{G}^{a}=\mathbf{Q}^{s}$. It is shown in Ref. 6 that the advection cell residual contributing to node $j$ from the element $E$ can be defined by the first-order N scheme as:

$$
\begin{equation*}
\boldsymbol{\Phi}_{j}^{N}=\boldsymbol{\Phi}_{j}^{a}+\mathbf{B}_{j}^{a} \boldsymbol{\Phi}^{s} \tag{9}
\end{equation*}
$$

where $\boldsymbol{\Phi}_{j}^{a}$ and $\boldsymbol{\Phi}_{j}^{s}$ are, respectively, the split cell-residual due to the advection and source terms:

$$
\begin{align*}
\mathbf{\Phi}_{j}^{a} & =-\mathbf{K}_{j}^{a^{+}}\left(\mathbf{U}_{j}-\tilde{\mathbf{U}}^{a}\right)  \tag{10}\\
\boldsymbol{\Phi}^{s} & =\frac{1}{3} \sum_{j \in E} \mathbf{Q}_{j}^{s} d \Omega^{E} \tag{11}
\end{align*}
$$

where

$$
\begin{equation*}
\mathbf{K}_{j}^{a}=\frac{1}{2} \mathbf{A}_{n_{j}}^{a}\left|\mathbf{n}_{j}\right|=\mathbf{K}_{j}^{a^{+}}+\mathbf{K}_{j}^{a^{-}}, \quad \tilde{\mathbf{U}}^{a}=\frac{\sum_{j \in E} \mathbf{K}_{j}^{a^{-}} \mathbf{U}_{j}}{\sum_{j \in E} \mathbf{K}_{j}^{a^{-}}}=-\frac{\sum_{j \in E} \mathbf{K}_{j}^{a^{-}} \mathbf{U}_{j}}{\sum_{j \in E} \mathbf{K}_{j}^{a^{+}}} \tag{12}
\end{equation*}
$$

and $\mathbf{B}_{j}^{a}$ is the Low-Diffusion-A (LDA) distribution matrix applied to the advection term: ${ }^{37}$

$$
\begin{equation*}
\mathbf{B}_{j}^{a}=\mathbf{K}_{j}^{a^{+}}\left(\sum_{i \in E} \mathbf{K}_{i}^{a^{+}}\right)^{-1} \tag{13}
\end{equation*}
$$

The positive and negative superscripts are the projections of the flux Jacobian onto the positive and negative characteristics. Note: $\sum_{j \in E} \mathbf{K}_{j}^{a}=0$. We now extend this known first-order N scheme to hyperbolic advection-diffusion system.

We follow the same procedure outlined above for advection system, and directly extend this advection scheme to our hyperbolic advection-diffusion system. The extension is possible with the non-unified technique proposed in Ref. 1, where we separate the wave structures of the advective and diffusive terms. Therefore, we only need to include the hyperbolic diffusion contribution to the split cell residual to Eq. (9) to extend the advection N scheme to hyperbolic advection-diffusion N scheme:

$$
\begin{equation*}
\boldsymbol{\Phi}_{j}^{N}=\boldsymbol{\Phi}_{j}^{a}+\boldsymbol{\Phi}_{j}^{d}+\mathbf{B}_{j}^{a} \boldsymbol{\Phi}^{s} \tag{14}
\end{equation*}
$$

where the split cell residual due to the hyperbolic diffusion term takes the following form

$$
\begin{equation*}
\mathbf{\Phi}_{j}^{d}=-\mathbf{K}_{j}^{d+}\left(\mathbf{U}_{j}-\tilde{\mathbf{U}}^{d}\right)+\frac{1}{3} \mathbf{B}_{j}^{\mathrm{d}} \sum_{i \in E} \mathbf{Q}_{i}^{d} d \Omega^{E} \tag{15}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathbf{K}_{j}^{d}=\frac{1}{2} \mathbf{A}_{n_{j}}^{d}\left|\mathbf{n}_{j}\right|=\mathbf{K}_{j}^{d^{+}}+\mathbf{K}_{j}^{d^{-}}, \quad \tilde{\mathbf{U}}^{d}=\frac{\sum_{j \in E} \mathbf{K}_{j}^{d^{-}} \mathbf{U}_{j}}{\sum_{j \in E} \mathbf{K}_{j}^{d-}}=-\frac{\sum_{j \in E} \mathbf{K}_{j}^{d^{-}} \mathbf{U}_{j}}{\sum_{j \in E} \mathbf{K}_{j}^{d^{+}}} \tag{16}
\end{equation*}
$$

and $\mathbf{B}_{j}^{d}$ is the LDA distribution matrix applied to the hyperbolic-diffusion term:

$$
\begin{equation*}
\mathbf{B}_{j}^{d}=\mathbf{K}_{j}^{d^{+}}\left(\sum_{i \in E} \mathbf{K}_{i}^{d^{+}}\right)^{-1} \tag{17}
\end{equation*}
$$

## B. Advection-diffusion Rusanov (Rv) scheme

The Rusanov scheme (aka Lax-Friedrichs) ${ }^{17}$ is obtained from a centered scheme with an added isotropic dissipation term. The Rusanov scheme is a positive and energy stable scheme. Our objective is to extend the Rusanov advection scheme to hyperbolic advection-diffusion system. The process is similar to the one we discussed in Sec. A; that is, start with the advection scheme and then add the contribution from the hyperbolic diffusion. According to the advection Rusanov scheme, the advection cell residual contributing to node $j$ from the element $E$ is:

$$
\begin{equation*}
\boldsymbol{\Phi}_{j}^{\mathrm{RV}}=\boldsymbol{\Phi}_{j}^{a}+\boldsymbol{\Phi}_{j}^{s}, \tag{18}
\end{equation*}
$$

where the split cell residual due to advection and source terms are

$$
\begin{align*}
\boldsymbol{\Phi}_{j}^{a} & =\frac{1}{3}\left(-\sum_{i \in E} \mathbf{K}_{i}^{a} \mathbf{U}_{i}\right)-\alpha^{a}\left(\mathbf{U}_{j}-\overline{\mathbf{U}}\right)  \tag{19}\\
\mathbf{\Phi}_{j}^{s} & =\mathbf{Q}_{j}^{s} \frac{d \Omega^{E}}{3} \tag{20}
\end{align*}
$$

where $\alpha^{a}$ is the spectral radius of the advection flux, defined as

$$
\begin{equation*}
\alpha^{a}=\max _{j \in E}\left(\left|\lambda^{a}\left(\bar{u}_{j}\right)\right|\right) \tag{21}
\end{equation*}
$$

We then follow the non-unified approach of Ref. 1 by splitting the advection and the hyperbolic diffusion wave structures, which allow us to extend the Rusanov advection scheme to hyperbolic advection diffusion scheme by simply adding the hyperbolic diffusion contribution to the split cell residual. We then arrive at the extended Rusanov hyperbolic advection-diffusion scheme:

$$
\begin{equation*}
\boldsymbol{\Phi}_{j}^{\mathrm{RV}}=\boldsymbol{\Phi}_{j}^{a}+\boldsymbol{\Phi}_{j}^{d}+\boldsymbol{\Phi}_{j}^{s} \tag{22}
\end{equation*}
$$

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where the hyperbolic diffusion contribution is defined as:

$$
\begin{equation*}
\boldsymbol{\Phi}_{j}^{d}=\frac{1}{3}\left(-\sum_{i \in E} \mathbf{K}_{i}^{d} \mathbf{U}_{i}\right)-\alpha^{d}\left(\mathbf{U}_{j}-\overline{\mathbf{U}}\right)+\mathbf{Q}_{j}^{d} \frac{d \Omega^{E}}{3} \tag{23}
\end{equation*}
$$

where the spectral radius associated with the hyperbolic diffusion flux is

$$
\begin{equation*}
\alpha^{d}=\max _{j \in E}\left(\sqrt{\nu\left(\bar{u}_{j}\right) / T_{r}\left(\bar{u}_{j}\right)}\right), \tag{25}
\end{equation*}
$$

with a note that the hyperbolic diffusion system has an isotropic wave structure; i.e., $\left|\lambda_{1}^{d}\right|=\left|\lambda_{2}^{d}\right|$.

## IV. High-order schemes

In this section we briefly discuss high-order hyperbolic RD schemes, which will be employed along with the extended first-order advection-diffusion schemes, for constructing shock-capturing schemes in Sec. V.

High-order RD schemes consist of a cell residual, $\boldsymbol{\Phi}^{E}$, and a distribution matrix, $\mathbf{B}_{j}$, such that $\sum_{j \in E} \mathbf{B}_{j}=$ $\mathbf{I}$, where $\mathbf{I}$ is an identity matrix. The nodal residual is then obtained as $\mathbf{B}_{j} \boldsymbol{\Phi}^{E}$. Various distribution matrices are available (see e.g., ${ }^{6,38}$ ), but here we focus on hyperbolic advection-diffusion schemes described in detail in Ref. 1; namely the baseline RD, the RD-CC2 and RD-CC3 schemes. The baseline RD scheme is the scheme of Ref. 2, and the second-order RD-CC2 and the third-order RD-CC3 schemes are designed such that the cell residual vanishes for exact quadratic and cubic functions, respectively. It is important to emphasize that these schemes are not merely a sum of advection and hyperbolic diffusion schemes, but rather hyperbolic diffusion terms are tightly coupled with the advection in the discrete level through curvature correction terms ( $\mathrm{see}^{1}$ for more detail).

For these high-order hyperbolic RD schemes, we use a SUPG distribution matrix, $\mathbf{B} j^{\text {SUPG }}$, which is obtained from the finite-element SUPG scheme ${ }^{39}$ in the RD framework. ${ }^{13}$ Briefly, the $\mathbf{B} j^{\text {SUPG }}$ consists of a Galekin term and a stabilization term. We can define advection and hyperbolic diffusion stabilization terms independently and arrive at the following distribution matrix: ${ }^{1}$

$$
\begin{equation*}
\mathbf{B}_{j}^{\mathrm{SUPG}}=\frac{1}{3} \mathbf{I}+(1-\omega) \mathbf{D}_{j}^{a}+\omega \mathbf{D}_{j}^{d} \tag{26}
\end{equation*}
$$

where $\omega$ is the weighting function defined as $2 /(R e+2)$, where $R e=\nu / a_{n}, a_{n}=\frac{\partial f}{\partial u} \hat{n}_{x}+\frac{\partial g}{\partial u} \hat{n}_{y}$, and $\mathbf{D}_{j}^{a}$ and $\mathbf{D}_{j}^{d}$ are the stabilization terms defined independently for the advective and the diffusive terms,

$$
\begin{align*}
& \mathbf{D}_{j}^{a}=\left[\begin{array}{ccc}
d_{j}^{\mathrm{SUPG}} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right], \quad d_{j}^{\mathrm{SUPG}}=\frac{1}{2} \frac{a_{n_{j}}\left|n_{j}\right|}{\sum_{i \in E} \max \left(0, a_{n_{i}}\right)\left|n_{i}\right|+\epsilon},  \tag{27}\\
& \mathbf{D}_{j}^{d}=\frac{1}{2} \mathbf{K}_{j}^{d}\left(\sum_{i \in E} \mathbf{K}_{i}^{d+}\right)^{-1}, \tag{28}
\end{align*}
$$

The $\epsilon \ll 1$ in the denominator of Eq. (27) is added to avoid division by zero when advection speed is identically zero.

In Ref. 1, we noted that the effects of weighting function is insignificant for smooth solution. However, in this study, we found it essential in obtaining discontinuous solutions, such as the nonlinear Burgers equation. Note that the weighting function is only applied to the stabilization terms and that $\sum_{j \in E} D_{j}^{a}=\sum_{j \in E} D_{j}^{d}=$ 0 .

## V. Blended schemes

In this section, we combine the extended first-order hyperbolic advection-diffusion scheme with high-order hyperbolic advection-diffusion RD schemes of Ref. 1, namely the baseline RD scheme, and RD-CC2 and RDCC3 schemes, which are intentionally designed to ensure that the cell residual vanishes for exact quadratic
and cubic functions, respectively. The RD-CC2 and RD-CC3 schemes are shown to produce accurate and smooth solution gradients on irregular triangular grids for smooth solutions. Here, we follow the approach of Deconinck and van der Weide, ${ }^{5}$ and extend these hyperbolic advection-diffusion schemes to discontinuous solutions through a nonlinear blending matrix $\Theta$ :

$$
\begin{equation*}
\boldsymbol{\Phi}_{j}^{\mathrm{B}}=\boldsymbol{\Theta} \boldsymbol{\Phi}_{j}^{\text {first-order }}+(\mathbf{I}-\boldsymbol{\Theta}) \boldsymbol{\Phi}_{j}^{\text {high-order }}, \tag{29}
\end{equation*}
$$

where $\boldsymbol{\Phi}_{j}^{\text {first-order }}$ is the extended first-order advection-diffusion scheme, and $\boldsymbol{\Phi}_{j}^{\text {high-order }}$ is either the RD baseline, the RD-CC2, or the RD-CC3 scheme. The blending matrix $\boldsymbol{\Theta}$ is defined as

$$
\boldsymbol{\Theta}=\left[\begin{array}{ccc}
\theta_{u} & 0 & 0  \tag{30}\\
0 & \theta_{p} & 0 \\
0 & 0 & \theta_{q}
\end{array}\right], \quad \theta_{r}=\frac{\left|\Phi_{r}^{E}\right|}{\sum_{j \in E}\left|\Phi_{r_{j}}^{\text {first-order }}\right|+\epsilon_{\theta}} \in[0,1]
$$

where $r=u, p, q$ (corresponding to the advection and the hyperbolic diffusion equations), the cell residual $\Phi_{r}^{E}$ is evaluated with either the RD baseline, the RD-CC2, or the RD-CC3 scheme, and $\epsilon_{\theta}$ is a constant introduced to avoid blending function to approach unity when both high-order and first-order cell residuals are vanishingly small. Here we have used $\epsilon_{\theta}=10^{-4}$. Note that the hyperbolic diffusion contribution of the cell residual is included both in the high-order and the first-order residual evaluations used in Eq. (30). We call the resulting blended schemes RD-B, RD-CC2-B and RD-CC3-B, respectively; i.e., appending a letter ' B ' to the high-order schemes. The proposed blended schemes are for an advection-diffusion system, and that itself is significant, ensuring that formal order of accuracy of high-order schemes is recovered in the smooth region.

## VI. Avoiding entropy-violating solutions (unphysical shocks)

Many shock-capturing schemes require an entropy fix to avoid unphysical shocks. In the RD method, there are few approaches in dealing with entropy-violating solutions. One approach is to modify the local wave speeds in the distribution matrix to break the symmetry that results in admitting unphysical shocks. This multi-dimensional entropy fix, proposed in Refs. 7, 27, is identical, in one-dimensional cases, to a reformulation of the conventional entropy fix of Harten and Hyman, ${ }^{40}$ which is a global technique meaning that the modification to the wave speed is applied to the entire elements in the computational domain. Thus, detecting different regions of the flow field is not needed.

A different approach was proposed in Refs. 25,26 based on a special quadrature formula for the cellresidual that recognizes physically-correct changes in the entropy. This special quadrature formula can be used globally, but to capture physical shocks precisely (assuming one of the cell faces is aligned with the shock) a different quadrature formula must be used in the shock region. The use of different quadrature formulas in two adjacent elements will, then, require a fix over the common edge for conservation, ${ }^{25,26}$ which is not very attractive as it makes the cell-residual evaluation non-local. It is possible to employ the entropy-satisfying cell-residual in all elements to avoid such an added complication as shock-aligned grids are not generally available anyway. However, entropy-satisfying quadrature formulas that preserve quadratic or cubic solutions, which are required in the RD-CC2 and RD-CC3 schemes, have yet to be analyzed, and therefore the method is not immediately applicable.

Here we propose an alternative approach in dealing with entropy-violating solutions. We realize that the first-order Rusanov scheme is formulated as a central scheme plus an isotropic dissipation. The central part could vanish for shocks, but the dissipation term can never vanish and thus the first-order Rusanov scheme cannot preserve any discontinuous solution, including expansion shocks. In addition, the very dissipative nature of the Rusanov scheme effectively spreads the sonic fan. Therefore, our proposed approach is based on activation of the first-order Rusanov scheme at the origin of the expansion shock. We will discuss this technique next, followed by a new characteristics-based nonlinear wave sensor to detect different regions of the flow field. The use of Rusanov scheme for sonic expansion is equivalent of adding the isotropic dissipation term, $\alpha\left(\mathbf{U}_{j}-\overline{\mathbf{U}}\right)$, to the baseline RD scheme at a sonic point. However, the use of the Rusanov scheme is more convenient than dealing with shocks by blending and with expansion shocks by an extra dissipation, because we can control shocks and expansion shocks in the same framework of the blended scheme with a single parameter $\boldsymbol{\Theta}$.

## A. Avoiding unphysical shocks by the first-order Rusanov scheme

Many schemes are susceptible to unphysical shocks (entropy-violating solutions) and require an entropy fix technique to avoid them. In many cases, unphysical shocks are captured because the cell-residual vanishes or equivalently preserves such solutions, which often arises as a side effect of an accurate shock capturing capability. One way to avoid them is, therefore, to design the cell-residual such that it may vanish for shock waves, but does not vanish for unphysical shocks as in Refs. 25,26. In this work, however, we employ a much simpler strategy, noting that the first-order Rusanov scheme cannot preserve both physical and unphysical shocks. Namely, we design the proposed blended schemes such that they reduce to the first-order Rusanov scheme at the origin of the sonic expansion, and thereby avoid capturing unphysical shocks. We accomplish this by setting the blending parameter associated with the advection equation to one; i.e., $\theta_{u}=1$. The proposed approach, therefore, requires accurate detection of the unphysical shocks. In the next section, we propose and explain in details a characteristics-based approach to accurately detect different regions of the flow field, including the sonic expansion regions.

## B. Characteristics-based nonlinear wave sensor

We propose a technique based on the steady characteristic waves to identify nonlinear waves: shocks, compression, and expansion regions within a domain. The proposed method is a great improvement over the technique reported in Refs. 25, 26 with minimal dependency on thresholding parameters and superiority in predicting various regions. The proposed approach may be used to mark the elements containing shock lines (or surfaces in 3D) and/or are within the compression or expansion regions. Initially, we used this information to construct a characteristics-based blending function, but observed that some thresholding parameter should be introduced to reduce the predicted overshoot and/or undershoot values around discontinuities. The proposed sensor is not just a shock sensor; it is a more general sensor that also detects various nonlinear waves such as compression and expansion waves. The proposed technique is also applicable in constructing a shock fitting technique, ${ }^{29,30}$ which will be discussed in future reports. A similar steady-characteristic-based shock sensor is proposed and extended to 3D flows in Ref. 41. A similar strategy may be taken to extend the sensor proposed here to 3D in future.

Consider an element $E$ with an area $d \Omega^{E}$, located either in the compression, expansion, or shock region (see Fig. 1). If we allow the vertices of the element to travel along the characteristic lines with the wave


Figure 1. Schematics of movements of an element $E$ with area $d \Omega^{E}$ along the steady characteristics lines in the expansion, compression, and shock regions.
velocity $\mathbf{w}$, we can obtain the rate of change of the element area as:

$$
\begin{equation*}
\frac{d}{d \tau} d \Omega^{E}=\operatorname{div}(\mathbf{w}) d \Omega^{E}=\frac{1}{2} \sum_{j \in E} \mathbf{w}_{j} \cdot \mathbf{n}_{j} \tag{31}
\end{equation*}
$$

As discussed in Refs. 25, 26, this quantity is negative if the element is inside a shock wave with converging characteristics, and therefore the area will vanish (and then go negative) for shocks. In Refs. 25, 26, the above quantity is directly employed with a simple normalization as a sensor for detecting shocks and sonic expansions. Here, we consider an alternative normalization based on the distance $D_{0}$ at which the element
travels with the steady characteristic velocity, w, until element area becomes zero. Suppose the time for an element to become an element with zero area (or volume in 3D) is $\tau_{0}$. Then, we can integrate Eq. (31) from $\tau=0$ to $\tau=\tau_{0}$ to get

$$
\begin{equation*}
0-d \Omega^{E}=\tau_{0} \frac{1}{2} \sum_{j \in E} \mathbf{w}_{j} \cdot \mathbf{n}_{j} \tag{32}
\end{equation*}
$$

Solving for $\tau_{0}$, we obtain

$$
\begin{equation*}
\tau_{0}=\frac{-2 d \Omega^{E}}{\sum_{j \in E} \mathbf{w}_{j} \cdot \mathbf{n}_{j}} \tag{33}
\end{equation*}
$$

which can be used to estimate the distance at which the element area vanishes:

$$
\begin{equation*}
D_{0}=\left|\tau_{0} \mathbf{v}\right|=\tau_{0}|\mathbf{v}|, \tag{34}
\end{equation*}
$$

where $\mathbf{v}$ is the average element velocity; i.e., $\mathbf{v}=1 / 3 \sum_{j \in E} \mathbf{w}_{j}$.
Consider now $\bar{h}$ as a distance across the element $E$ taken by the averaged characteristic velocity $\mathbf{v}$. That is:

$$
\begin{equation*}
\bar{h}=\left|\mathbf{x}_{\text {out }}-\mathbf{x}_{\text {in }}\right| \tag{35}
\end{equation*}
$$

where $\mathbf{x}_{\text {in }}$ and $\mathbf{x}_{\text {out }}$ are the coordinates of the inflow and outflow as depicted in Fig. 2, which can be evaluated

(a) one-target case

(b) two-target case

Figure 2. Schematic of one- and two-target cases, corresponding to one and two upwind nodes, in defining $\bar{h}$.
(similar to the LDA technique) as

$$
\begin{equation*}
\mathbf{x}_{\text {in }}=\frac{\sum_{j \in E} k_{j}^{-} \mathbf{x}_{j}}{\sum_{j \in E} k_{j}^{-}}, \quad \mathbf{x}_{o u t}=\frac{\sum_{j \in E} k_{j}^{+} \mathbf{x}_{j}}{\sum_{j \in E} k_{j}^{+}} \tag{36}
\end{equation*}
$$

where

$$
\begin{equation*}
k_{j}^{+}=\max \left(0, k_{j}\right), \quad k_{j}^{-}=\min \left(0, k_{j}\right), \quad k_{j}=\frac{1}{2} \mathbf{v} \cdot \mathbf{n}_{j} . \tag{37}
\end{equation*}
$$

Substituting Eq. (36) into Eq. (35), we arrive, after some algebra, at the following formulation:

$$
\begin{equation*}
\bar{h}=\frac{d \Omega^{E}}{\sum_{j \in E} k_{j}^{+}}|\mathbf{v}| \tag{38}
\end{equation*}
$$

where we have noted that $k_{j}=k_{j}^{-}+k_{j}^{+}$and $\sum_{j \in E} n_{x_{j}} x_{j}=\sum_{j \in E} n_{y_{j}} y_{j}=2 d \Omega^{E}$.
Comparing the distance $D_{0}$ with the distance $\bar{h}$, which is the distance across the element taken by the averaged characteristic velocity, $\mathbf{v}$, we propose the following criteria for various regions of the flow field according to the value of $\alpha=\bar{h} / D_{0}$ ratio:

$$
\alpha=\frac{-\sum_{j \in E} \mathbf{w}_{j} \cdot \mathbf{n}_{j}}{2 \sum_{j \in E} k_{j}^{+}}=\left\{\begin{array}{cl}
\geq 1 & \text { shock }  \tag{39}\\
0<\ldots<1 & \text { compression } \\
-1<\ldots \leq 0 & \text { expansion } \\
\leq-1 & \text { sonic expansion }
\end{array}\right.
$$

Figure 3 shows computed $\alpha$ sensor within the entire domain from a sample solution of a nonlinear viscous Burgers equation with $\nu=10^{-6}$ on a $64 \times 64$ irregular grid $\in(0,1.5) \times(0,1)$ with $u(x, 0)=1.5-2 x$ as a boundary condition.

We now summarize the discussion with a step-by-step procedure for a general advection equation (i.e., $\left.u_{t}+f_{x}+g_{y}=0\right)$ as following:


Figure 3. Elemental value of $\alpha$ sensor computed for a sample nonlinear viscous Burgers equation with $64 \times 64$ irregular grids.

- loop over elements
- evaluate $\mathbf{w}=\left(f_{u}, g_{u}\right)$ for every vertex. For example, for Burgers equation, we have $\mathbf{w}=(u, 1)$
- evaluate average characteristic velocity of the element $\mathbf{v}=1 / 3\left(\sum_{j}\left(f_{u}\right)_{j}, \sum_{j}\left(g_{u}\right)_{j}\right)$. For Burgers equation, we get $\mathbf{v}=\left(1 / 3 \sum_{j} u_{j}, 1\right)$
- evaluate $k_{j}^{+}=1 / 2 \max \left(0, \mathbf{v} \cdot\left(n_{x_{j}}, n_{y_{j}}\right)\right)$
- evaluate $\sum_{j} \mathbf{w}_{j} \cdot\left(n_{x_{j}}, n_{y_{j}}\right)$
- evaluate $\alpha$ as given in Eq. (39)
- set $\theta_{u}=1.0$ for $\alpha \leq-0.9$ to activate the Rusanov scheme for avoiding unphysical shocks; note that in Eq. (39) $\alpha \leq-1.0$ corresponds to the sonic expansion region but the actual computed $\alpha$ value, due to some numerical errors, may not be precisely $\leq-1.0$. Therefore, we propose $\alpha \leq-0.9$. We also note that the entropy-violating phenomena is an advection problem and therefore, we do not modify the blending parameters associated with the hyperbolic diffusion terms (i.e., $\theta_{p}$ and $\theta_{q}$ ); modification to the blending parameters $\theta_{p}$ and $\theta_{q}$ will destroy the accuracy of the predicted solution gradients.

The above procedure is used as a part of the shock-capturing hyperbolic blended schemes discussed in Sec. V.

## VII. Boundary condition

The details of formulating strong boundary condition for a hyperbolic residual-distribution scheme are reported in Ref. 1. Imposing a strong BC is not possible for some boundaries and, instead, a weak formulation needs to be constructed. The weak boundary condition is already developed and used within the RD community. $7,18,19$ Here, we follow the same procedure that is already employed by the RD community to construct a weak outflow boundary condition, which is the only weak boundary condition used in this study, for hyperbolic adection-diffusion schemes.

## A. Weak outflow boundary condition

Following the procedure widely used within the RD community (see e.g., Ref. 7) we start the process by creating as many virtual ghost elements as boundary nodes. This is done by adding a node in an opposite direction of the boundary face normals, such that the ghost element is an element with a 90 -degree angle


Figure 4. Schematic of the weak boundary condition process using ghost elements. Steps 2 and 3 are illustrated for the left node. The same is repeated for the right node as illustrated in Step 1.
as shown in Fig. 4a. Depending on the boundary type, we can use different values in the ghost nodes. For the outflow boundary condition, which is the only weak BC we studied here, we simply use the same information that is available from the vertex of the boundary element that is not on the boundary. This step is schematically shown in Fig. 4b:

$$
\begin{equation*}
\mathbf{U}_{g}=\mathbf{U}_{j} \tag{40}
\end{equation*}
$$

where subscript $g$ denotes the ghost vertex. We then formulate the extended N scheme described in Sec. A while simultaneously allowing vertex $g$ to approach vertex $b$, which is the boundary node, to form a zerovolume ghost cell (see Fig. 4c). The ghost cell contribution to the boundary node residual, after some algebra, becomes

$$
\begin{equation*}
\boldsymbol{\Phi}_{b}=-\left((1-\omega) \mathbf{K}_{b}^{a+}+\omega \mathbf{K}_{b}^{d+}\right)\left(\mathbf{U}_{b}-\mathbf{U}_{g}\right) \tag{41}
\end{equation*}
$$

where we have incorporated a weighting function $\omega=2 /(R e+2), R e=\nu / a_{n}$ to emphasize the relative importance of advection and diffusion, and used the geometrical relation $\mathbf{n}_{g}=-\mathbf{n}_{b}$. This weak boundary formulation is upwinding due to the presence of $\mathbf{K}^{+}$. We verify this with an example in Sec. VIII. In our experience, the weighting has been found important to achieve greater residual convergence in iterative solvers.

We then add the ghost cell residual contribution from the boundary nodes, $\boldsymbol{\Phi}_{b}$, to the boundary nodal residuals computed from the interior schemes. The corresponding Jacobian contribution, obtained with the use of Automatic Differentiation through an operator overloading technique using chain rules, is also added to the Jacobian computed from the interior nodes.

## VIII. Results

All examples presented in this section are solved on a series of irregular and perturbed grids, unless otherwise stated. A representative of an irregular mesh is shown in Fig. 5. The implicit solver as described in Ref. 1 is used to solve the nodal-residual equations. The linear relaxation is performed with a Gauss-Seidel algorithm to reduce the linear residuals by two orders of magnitude with a maximum of 1000 relaxation steps (Note: we do not claim that is an optimal setting). The implicit solver is considered to be converged when ten orders of magnitude residual reduction is obtained for all the equations. To avoid instability with the implicit solver, full Newton update is not allowed at the beginning of the simulations (typically 20-30 Newton iterations). After the initial steps, a full Newton update is performed, which results in a converged solution typically in 5-15 additional Newton iterations.

In this section, we seek the following objectives, which are addressed in the next subsections:

- the hyperbolic formulation of viscous terms do not negatively affect the solution of the inviscid equation as the viscosity coefficient approaches zero,
- a weak outflow boundary condition, constructed for hyperbolic advection-diffusion schemes, is implemented correctly,
- the proposed blended hyperbolic RD schemes can accurately capture discontinuity,


Figure 5. A representative of a perturbed and irregular grid used in all the presented examples (unless otherwise stated).

- non-physical shocks (e.g., sonic expansion) can be avoided by the proposed non-linear wave sensor technique,
- the formal order of accuracy of the proposed blended schemes are preserved in smooth regions.


## A. Hyperbolic Advection-Diffusion System vs. Scalar Advection

Consider the following nonlinear viscous Burgers equation

$$
\begin{align*}
\partial_{x}\left(\frac{u^{2}}{2}\right)+\partial_{y} u & =\nu\left(\partial_{x x} u+\partial_{y y} u\right)  \tag{42}\\
u(x, y) & =1.5-2 x, \quad \text { on } y=0 \tag{43}
\end{align*}
$$

The problem has the following exact solution in $(x, y) \in[0,1] \times[0,1]$ as $\nu \rightarrow 0$ :

$$
u(x, y)= \begin{cases}\text { for } y \geq 0.5, & \begin{cases}-0.5, & \text { if }-2(x-0.75)+(y-0.5) \leq 0 \\ 1.5, & \text { else },\end{cases}  \tag{44}\\ \text { otherwise, } & \max \left(-0.5, \min \left(1.5, \frac{x-0.75}{y-0.5}\right)\right)\end{cases}
$$

We solve the above equation using scalar advection schemes (N, Rusanov, and the baseline RD with SUPG distribution) and compare them with the solutions of the hyperbolic advection-diffusion system formulation for vanishingly small diffusion coefficient $\left(\nu=10^{-6}\right)$. To remove the effects of weak outflow boundary condition formulation (if any) and better characterize the hyperbolic system formulation against the scalar advection schemes, we employed a strong boundary condition. ${ }^{1}$ The results of this exercise are presented in Fig. 6, which shows that the hyperbolic advection-diffusion system formulation with vanishingly small diffusion coefficient (shown as Hyp-ADE) produces almost identical results as the scalar advection schemes. These results verify that the hyperbolic advection-diffusion formulation does not affect the solution of the inviscid solution as viscosity coefficient approches zero. Therefore, only the hyperbolic advection-diffusion system formulation is presented in the following examples. In this example, the blending formulation is not used and thus, under/overshoot values around the discontinuity, which are present in both scalar advection and hyperbolic system results, are expected with the baseline RD scheme.

## B. Verification of Weak Outflow BC

This example is presented to verify the weak outflow BC for hyperbolic advection-diffusion schemes.


Figure 6. Comparison between the scalar advection schemes (first row), and the non-blended hyperbolic advection-diffusion (Hyp-ADE) schemes with $\nu=1 \times 10^{-6}$ (second row) for the Burgers problem in ( $x, y$ ) $\in$ $[0,1] \times[0,1]$ with a $64 \times 64$ irregular and perturbed grid.

Consider the nonlinear viscous Burgers equation, Eq. (42), along with the boundary condition given in Eq. (43) for $(x, y) \in[0,1.5] \times\left[0, y_{\max }\right]$, which results in a formation of a normal shock along the $x=0.75$ plane. Here, three $y_{\max }$ values of $0.4,0.6$, and 1.0 are considered. A $64 \times 64$ irregular and perturbed triangular grid was generated for the largest domain (i.e., $y_{\max }=1.0$ ) as the basis mesh. Then, smaller grid sizes for the other two domains are generated such that the number of grid points in the y-direction becomes approximately proportional to the $y_{\max }$ value. For example, a $64 \times 25$ grid is generated for the domain with $y_{\max }=0.4$ $(64 \times 0.4 \sim 25)$. Similarly, a grid size of $64 \times 38$ is generated for the domain with $y_{\max }=0.6(64 \times 0.6 \sim 38)$. This is to ensure that the truncated solution can be produced with the corresponding truncated grid system, verifying the accuracy of the weak outflow boundary condition. We use the hyperbolic advection-diffusion system and solve the above problem using the baseline RD scheme with the proposed weak outflow boundary condition applied at $y=y_{\max }$ as shown in Fig. 7.


Figure 7. Verification of the weak outflow BC formulation for the hyperbolic RD scheme. Solutions are for the viscous Burgers problem with irregular and perturbed grids.

## C. Blended Hyperbolic Advection-Diffusion Schemes

IIn this example, we compare solutions of the viscous Burgers equation, Eq. (42), along with the boundary condition given in Eq. (43) that are predicted with the proposed blended hyperbolic advection-diffusion schemes. The blended schemes are based on the presented first-order N and Rusanov (Rv) schemes (see Sec. III). Figure 8 shows solutions obtained with the blended baseline RD, RD-CC2 and RD-CC3 schemes, called, respectively, RD-B, RD-CC2-B, and RD-CC3-B. The results show that the proposed blended hyperbolic schemes accurately predict both the compression waves and the discontinuity.

The predicted solution with the proposed Rusanov-based blended schemes across the shock at $y=0.6$ as well as the solution and the solution gradients across the compression region at $y=0.3$ are compared with the corresponding exact values in Fig. 9. The solution across the shock is accurately detected with no oscillation. The accuracy of the RD-CC2-B and the RD-CC3-B schemes are more evident in the compression region. The proposed blended RD-CC2-B scheme predicts a solution and solution gradients that are significantly more accurate than the baseline RD-B scheme. Even though the second-order RD-CC2 scheme approaches


Figure 8. Comparison between the proposed $N$-based (first-row) and Rusanov-based (second-row) blended hyperbolic RD schemes applied to the Burgers problem in $(x, y) \in[0,1] \times[0,1]$ on $100 \times 100$ irregular and perturbed grid.


Figure 9. Comparison between the proposed Rusanov-based blended hyperbolic RD schemes for the Burgers problem in $(x, y) \in[0,1] \times[0,1]$ on $100 \times 100$ irregular and perturbed grids. First-row: $u(x, 0.6)$, second-row: $u(x, 0.3)$, third-row: $u_{x}(x, 0.3)$, forth-row: $u_{y}(x, 0.3)$.


Figure 10. Comparison between the proposed N-based and Rusanov-based blended hyperbolic RD-CC3 scheme for the Burgers problem in $(x, y) \in[0,1] \times[0,1]$ on $100 \times 100$ irregular and perturbed grids. First-row: $u(x, 0.6)$, second-row: $u(x, 0.3)$, third-row: $u_{x}(x, 0.3)$, forth-row: $u_{y}(x, 0.3)$.
the third-order accurate RD-CC3 solution for small viscosity coefficient as shown in Ref. 1, the improved solution gradients predicted by the blended RD-CC3-B scheme is remarkably noticeable. For comparison, reconstructed solution gradients using quadratic least-squares are also shown for $u_{x}$. The reconstructed solution gradient in $y$-direction was completely off the chart and therefore not shown. The reconstructed solution gradients are very oscillatory and inaccurate even with a high-order (in this case, third-order) solution. The predicted solution gradients are significantly better but also show relatively small oscillations across the compression fans, where the solution is continuous but the gradients are discontinuous. The Rusanov-based blended schemes also performed slightly better than the N-based blended schemes (Fig. 10).


Figure 11. Elemental residual and blending indicators based on the proposed blended hyperbolic RD schemes for the Burgers problem in $(x, y) \in[0,1] \times[0,1]$ on $100 \times 100$ irregular and perturbed grids (Note: Nodal residuals are converged to less than $10^{-10}$ ).

The converged solutions are obtained by freezing the blending parameter, $\boldsymbol{\Theta}$, after a certain number of Newton iterations, typically $30-35$. Continuous changes to $\Theta$ causes the residuals to plateau. This can be remedied by freezing the blending parameter. Similar practice is also used with conventional RD schemes. ${ }^{12}$

Contour plots of the cell residual and the blending parameters, $\boldsymbol{\Theta}$, are shown in Fig. 11 along with the exact solution, which is over-plotted with the predicted solution obtained on a $100 \times 100$ irregular and perturbed grid. The map of the blending parameter $\theta_{u}$ also indicates that only small number of elements are in fact blended with the first-order hyperbolic scheme, and the cell residual is very small $\left(<10^{-7}\right)$ throughout smooth regions. This is remarkable because other works reported by the RD community presented a blending map that covers almost the majority of the domain, including the compression region, with a large $\theta$ value close to unity (e.g., see Ref. 7). Note, reducing the $\epsilon_{\theta}$ given in Eq. (30) increases the $\theta_{u}$ value in the smooth region.

## D. Sonic Expansion

In this exercise, we examine the capability of the proposed blended schemes in accurately predicting a sonic expansion. Consider a nonlinear viscous Burgers equation, Eq. (42), with the following boundary condition

$$
u(x, y)= \begin{cases}-1.0, & x \leq 0  \tag{45}\\ +1.0, & \text { else }\end{cases}
$$

and $u=0$ as an initial solution. We first verify that the Rusanov scheme, unlike the baseline second-order RD scheme, does not require an entropy fix even on a regular grid (see Fig. 12). On the other hand, the Rusanovbased blended RD scheme, RD-B, predicts unphysical shocks on regular grid. This is because the the cell residual vanishes (resulting $\boldsymbol{\Theta} \rightarrow \mathbf{0}$ ) in the cells containing the discontinuity and thus, no signal is being sent to the corresponding nodes. As discussed in Refs. 25, 26, the cell residual evaluated by the Trapezoidal rule has a property of recognizing discontinuous solutions: it vanishes over an element having a side aligned with a shock. However, it cannot distinguish physical and non-physical shocks, and thus it vanishes also for expansion shocks. This shortfall can be avoided with an alternative quadrature formula proposed in Refs. 25,26 . However, the improved second-order RD-CC2 and the third-order RD-CC3 schemes, ${ }^{1}$ which are used in the construction of the proposed high-order blended schemes (i.e., RD-CC2-B and RD-CC3-B), are constructed based on the curvature correction terms ${ }^{1}$ and thus, are not precisely based on the trapezoidal rule. That is, these schemes behave much better in avoiding unphysical shocks even without an entropy-fix (see Fig. 12).

On irregular grids, the situation is slightly better, even for the baseline RD because the element at the sonic point does not have an edge perfectly aligned with a potential expansion shock: the sonic expansion occurs in a region of regular grids (see Fig. 13).

We can further improve the sonic expansion predictions with the help of the proposed characteristicsbased nonlinear wave sensor, $\alpha$, which can accurately detect the expansion region (see Sec. A). With the detection of elements in the origin of the expansion, we employ the extended first-order advection-diffusion Rusanov scheme (see Sec. B) by setting $\theta_{u}=1.0$ when $\alpha \leq-0.9$. These results are shown in Figs. 14 and 15 , respectively, for regular and irregular grids. A map of the $\alpha$ sensor is also shown for each of the schemes, confirming that the expansion fan is accurately predicted.

## E. Accuracy Verification

n this example, we consider a linear advection-diffusion equation with an exact smooth exponential solution of the form: ${ }^{42}$

$$
\begin{equation*}
u(x, y)=C \cos (A \pi \eta) \exp \left(\frac{1-\sqrt{1+4 A^{2} \pi^{2} \nu^{2}}}{2 \nu} \xi\right) \tag{46}
\end{equation*}
$$

where $A$ and $C$ are arbitrary constants, $\xi=a x+b y$, and $\eta=b x-a y$. We solve the system using the proposed blended schemes on a series of irregular and perturbed triangular grids. The order of accuracy of the proposed blended schemes are shown in Fig. 16. These results confirm that the blended RD-CC2-B scheme remains second-order for all the variables, while a uniform fourth-order solution is obtained with the blended RD-CC3-B scheme. Note that as we explained in Ref. 1, our design principle (preserving exact cubic solution) is not a necessary condition, and rather is a sufficient condition to guarantee a third-order accurate solution and solution gradients. Therefore, without a mathematical proof, we can only state that the scheme is at least third-order accurate. The baseline blended RD-B scheme, similar to the baseline RD scheme, predicts nearly second-order accurate solution gradients but, as shown in Ref. 1, the predicted gradients are extremely noisy even for smooth solutions. Thus, we recommend the blended RD-CC2-B and RD-CC3-B schemes for uniform second- and third-order order accurate solution and solution gradients.

## IX. Conclusions

We have developed second- and third-order shock-capturing blended hyperbolic RD schemes for advectiondiffusion problems on irregular triangular grids. We extended the first-order advection N and Rusanov schemes to hyperbolic advection-diffusion problems, and and constructed high-order blended hyperbolic RD


Figure 12. Sonic expansion prediction comparison between the non-blended RD, the first-order Rusanov, and the proposed baseline and high-order blended schemes on $64 \times 64$ regular grid without activation of the $\alpha$ sensor ( $\nu=0.001$ ).


Figure 13. Sonic expansion prediction comparison between the non-blended RD, the first-order Rusanov, and the proposed baseline and high-order blended schemes on $64 \times 64$ irregular and perturbed grid without activation of the $\alpha$ sensor $(\nu=0.001)$.


Figure 14. Sonic expansion prediction comparison between the baselined and the proposed blended schemes using $64 \times 64$ regular grid with activation of the $\alpha$ sensor as proposed in Sec. VI ( $\nu=0.001$ ).


Figure 15. Sonic expansion prediction comparison between the baselined and the proposed blended schemes using $64 \times 64$ irregular and perturbed grid with activation of the $\alpha$ sensor as proposed in Sec. VI ( $\nu=0.001$ ).


Figure 16. Accuracy of the proposed blended schemes for the smooth exponential solution ( $A=2, C=-0.009$, $a=2.0, b=1.0, \nu=0.01$ ).
schemes with a nonlinear blending function. We showed that the solution of the hyperbolic advectiondiffusion system approaches the solution of the advection problem as the diffusion coefficient approaches zero, confirming that the hyperbolic diffusion formulation does not negatively affect the advection scheme. We examined the blended hyperbolic RD schemes for discontinuous problems, and showed that the developed blended schemes can accurately predict discontinuous solutions with no oscillation. We proposed a new characteristics-based nonlinear wave sensor to detect different regions of the domain (e.g., shocks, expansion, compression), and demonstrated that the blended hyperbolic RD schemes do not suffer from entropy-violating solutions particularly when the proposed nonlinear wave sensor is used to activate the extended Rusanov scheme at the origin of the sonic expansion region. The high-order accuracy of the blended schemes was further verified on a series of irregular and perturbed grids.

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