Towards Future Navier-Stokes Schemes: Uniform Accuracy, O(h) Time Step, and Accurate Viscous/Heat Fluxes

Hiroaki Nishikawa*

National Institute of Aerospace, Hampton, VA, 23666, USA

We report progress towards the development of Navier-Stokes schemes having the following features: uniform accuracy with a compact stencil for all Reynolds numbers, O(h)time step for all Reynolds numbers through the viscous limit, and simultaneous computation of diffusive fluxes (viscous stresses/heat fluxes) at the same order of accuracy as the main variables. In this paper, we show that various such schemes can actually be constructed for the advection-diffusion equation, in the form of finite-difference, finite-volume, or residual-distribution, based on a first-order hyperbolic advection-diffusion system, which is an extended system of the first-order hyperbolic diffusion system introduced in Ref.[1]. Following the study on the model problem, we discuss its extension to the Navier-Stokes equations and potential benefits of Navier-Stokes schemes that will emerge as a result.

I. Introduction

It all started from a simple idea:

the diffusion equation is equivalent to a hyperbolic relaxation system, independently of the relaxation time, at a steady state.

In Ref.[1], based on this idea, we developed second-order residual-distribution schemes for a hyperbolic relaxation system (the first-order hyperbolic diffusion system), deriving an optimal (non-physical) relaxation time for a rapid convergence to a steady state, and demonstrated remarkable properties of the proposed approach: large O(h) time step and accurate solution gradients. In particular, we showed that simple explicit schemes applied to the first-order system (i.e., simple Jacobi iterations) converged to a steady state with O(N) iterations in one dimension and $O(N^{\frac{1}{2}})$ iterations in two dimensions, where N is the number of unknowns.

In a subsequent paper[2], we extend the first-order system approach to advection-diffusion problems. The extension is simple and straightforward: add an advection term to the first-order hyperbolic diffusion system and apply an upwind scheme for the whole hyperbolic advection-diffusion system. In this way, we no longer need to develop diffusion and advection schemes separately and combine them; we simply develop a single scheme for the hyperbolic advection-diffusion system. In one dimension, we develop a three-point finite-difference (or finite-volume or residual-distribution) scheme, which is capable of producing second-order accurate solution and gradient at a steady state on arbitrary grids. In two dimensions, we develop a finite-difference scheme on structured (non-uniform) grids, and also a multidimensional upwind scheme on unstructured triangular grids. We demonstrate that these schemes converge very rapidly to a steady state with O(h) time step, produce accurate solution gradients (diffusive fluxes), and preserve the design accuracy, for all Reynolds numbers.

Following the fundamental development, we discuss possible extensions of the proposed approach to the Navier-Stokes equations, for developing Navier-Stokes schemes which are uniformly accurate on a compact stencil for all Reynolds numbers, stable with O(h) time step through the viscous limit, and capable of simultaneously computing diffusive fluxes (viscous stresses/heat fluxes) at the same order of accuracy as the main variables.

^{*}Research Scientist, National Institute of Aerospace, 100 Exploration Way, Hampton VA 23666, AIAA Member.

II. Non-Stiff Hyperbolic Diffusion System

It is well known[3–5] that the diffusion equation,

$$u_t = \nu u_{xx}, \quad \nu > 0, \tag{1}$$

is asymptotically equivalent to the following first-order system (the hyperbolic heat equations):

$$u_t = \nu p_x,$$

$$p_t = (u_x - p)/T_r,$$
(2)

where T_r may be called a relaxation time, in the limit, $T_r \to 0$. There have been many attempts to develop numerical methods for such relaxation systems [6–8], with a particular focus on the stiff source term: an explicit time step, $\Delta t = O(T_r) \to 0$, is prohibitively restricted due to the extremely small relaxation time; an implicit treatment of the stiff source term could degrade the solution accuracy[9]. Although based on the same equations, our approach is different from these relaxation methods. The key is to realize the fact that the first-order system (2) is equivalent to the diffusion equation (1) at a steady state ($u_t = p_t = 0$) for *arbitrary* T_r :

Then, as far as the steady state computation is concerned, T_r is a free parameter; it can be chosen, for example, to reach the steady state as quickly as possible[1]. The stiffness is, thus, no longer an issue. For time-marching steady state computations, we find it advantageous to use this *non-stiff* first-order diffusion system. First, explicit time-stepping schemes can be used with a large O(h) time step (not $O(h^2)$ which is typical to diffusion). Second, the solution gradient, p, can be computed simultaneously with the same order of accuracy as u. Third, upwind schemes and various related techniques (e.g., limiters, preconditioning, Riemann solvers, etc) are all directly applicable to diffusion because it is now expressed as a hyperbolic system. In Ref.[1], we demonstrated these advantages for a pure diffusion problem, developing second-order Lax-Wendroff-type schemes for a uniform one-dimensional grid as well as triangular grids. It is also shown in Ref. [1] that the standard Galerkin discretization (continuous piecewise linear basis) can be derived as a special case of the proposed scheme. In effect, the first-order system approach may be considered as a way of deriving various diffusion schemes via advection schemes. It can also be used to derive various scalar diffusion schemes simply by ignoring updates to the gradient variable (using explicit gradient reconstruction instead)[1, 10]. See Ref.[1] for more details on diffusion problems. Here, we focus on its extension to advection-diffusion problems.

III. Unification of Advection and Diffusion

Consider the one-dimensional advection-diffusion problem,

$$u_t + au_x = \nu \, u_{xx} \quad \text{in } \Omega = (0, 1),$$
(4)

where u(0) and u(1) are given as boundary conditions, a is a positive advection speed. To compute the steady state solution to this problem, we propose to solve the first-order hyperbolic advection-diffusion system:

$$\mathbf{U}_t + \mathbf{A}\mathbf{U}_x = \mathbf{Q},\tag{5}$$

where

$$\mathbf{U} = \begin{bmatrix} u \\ p \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} a & -\nu \\ -1/T_r & 0 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} 0 \\ -p/T_r \end{bmatrix}, \quad (6)$$

where T_r is a free parameter. We determine T_r by requiring it to be comparable to the characteristic time scale, resulting a simple formula [2],

$$T_r = \frac{L_r}{a + \frac{\nu}{L_r}},\tag{7}$$



Figure 1. Plot of the optimal L_r (8). $L_r^{adv} = \frac{1}{\pi}$ and $L_r^{diff} = \frac{1}{\sqrt{2\pi}}$ are the limiting values.

where L_r is a length scale, which may be determined to further enhance the convergence to the steady state; the optimal formula has been derived in [2] as

$$L_r = \frac{1}{2\pi} \left[\frac{Re_{\pi}}{\sqrt{1 + Re_{\pi}^2} + 1} + \sqrt{1 + \frac{2}{\sqrt{1 + Re_{\pi}^2} + 1}} \right], \quad Re_{\pi} \equiv \frac{a(1/\pi)}{\nu}.$$
 (8)

(See Figure 1. Simply taking $L_r = 1/\pi$ also works well. Alternatively, it may be chosen to optimize the smoothing property for use in multigrid[1]. See Ref.[2] for a complete account of T_r and L_r .) This defines the first-order advection-diffusion system completely. Note that advection and diffusion have just been unified into a single hyperbolic system. The Jacobian matrix, **A**, has real eigenvalues:

$$\lambda_1 = -\frac{a}{Re_{L_r}}, \quad \lambda_2 = a\left(1 + \frac{1}{Re_{L_r}}\right),\tag{9}$$

where

$$Re_{L_r} \equiv \frac{aL_r}{\nu}.$$
(10)

Observe that as $Re_{L_r} \to 0$, the eigenvalues reduce to the diffusion characteristic speeds (denoted by a_d):

$$a_d \equiv \pm \frac{\nu}{L_r},\tag{11}$$

which is *finite* since $L_r = O(1)$, while as $Re_{L_r} \to \infty$, they approach 0 and a, implying scalar advection. In fact, the Reynolds number, Re_{L_r} , is the ratio of the pure advection and diffusion speeds:

$$Re_{L_r} = \frac{aL_r}{\nu} = \frac{a}{\nu/L_r} = \frac{a}{|a_d|}.$$
(12)

This is the key dimensionless parameter in the first-order advection-diffusion system that describes the balance between advection and diffusion.

To gain further insight, consider the decomposition of A:

$$\mathbf{A} = \mathbf{R}\mathbf{\Lambda}\mathbf{L} = \lambda_1 \mathbf{\Pi}_1 + \lambda_2 \mathbf{\Pi}_2,\tag{13}$$

where

$$\mathbf{\Pi}_{1} = \mathbf{r}_{1}\boldsymbol{\ell}_{1} = \begin{bmatrix} \frac{1}{Re_{L_{r}}+2} & \frac{L_{r}}{Re_{L_{r}}+2} \\ \frac{Re_{L_{r}}+1}{Re_{L_{r}}+2} \frac{1}{L_{r}} & \frac{Re_{L_{r}}+1}{Re_{L_{r}}+2} \end{bmatrix},$$
(14)

$$\mathbf{\Pi}_{2} = \mathbf{r}_{2}\boldsymbol{\ell}_{2} = \begin{bmatrix} \frac{Re_{L_{r}} + 1}{Re_{L_{r}} + 2} & \frac{-L_{r}}{Re_{L_{r}} + 2} \\ -\frac{Re_{L_{r}} + 1}{Re_{L_{r}} + 2} \frac{1}{L_{r}} & \frac{1}{Re_{L_{r}} + 2} \end{bmatrix},$$
(15)

and \mathbf{r}_k and ℓ_k are the k-th column of \mathbf{R} (k-th right-eigenvector) and the k-th row of \mathbf{L} (k-th left-eigenvector) respectively. The matrices, Π_1 and Π_2 , are the projection matrices that project the system (or a solution change) onto the corresponding subspaces: the left-running and right-running waves, respectively. It is interesting that these projection matrices can be expressed as, for both i = 1 and 2,

$$\mathbf{\Pi}_{i} = \frac{Re_{L_{r}}}{Re_{L_{r}} + 2} \,\mathbf{\Pi}_{i}^{A} + \frac{2}{Re_{L_{r}} + 2} \,\mathbf{\Pi}_{i}^{D},\tag{16}$$

where $\mathbf{\Pi}_{i}^{A}$ and $\mathbf{\Pi}_{i}^{D}$ are the projection matrices in the advection and diffusion limits:

$$\mathbf{\Pi}_{i}^{A} = \lim_{Re_{L_{r}} \to \infty} \mathbf{\Pi}_{i}, \quad \mathbf{\Pi}_{i}^{D} = \lim_{Re_{L_{r}} \to 0} \mathbf{\Pi}_{i}.$$
(17)

This means that each subspace is a linear combination of its own limits: pure advection and diffusion. This suggests then that we may construct an advection-diffusion scheme by combining a pure advection scheme and a pure diffusion scheme by using the weights as in (16). Here, we rather consider developing a scheme for the entire advection-diffusion system (5); such a linear combination emerges as a result.

For two-dimensional problems, the first-order advection-diffusion system is given by

$$u_t + au_x + bu_y = \nu (p_x + q_y),$$

$$p_t = (u_x - p)/T_r,$$

$$q_t = (u_y - q)/T_r.$$
(18)

Variables, p and q, are the additional variables which will be equivalent to the solution gradients (diffusive fluxes), u_x and u_y , respectively, at a steady state. In the vector form, the system (18) is written as

$$\mathbf{U}_t + \mathbf{A}\mathbf{U}_x + \mathbf{B}\mathbf{U}_y = \mathbf{Q} \tag{19}$$

where

$$\mathbf{U} = \begin{bmatrix} u \\ p \\ q \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} a & -\nu & 0 \\ -1/T_r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b & 0 & -\nu \\ 0 & 0 & 0 \\ -1/T_r & 0 & 0 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} 0 \\ -p/T_r \\ -q/T_r \end{bmatrix}.$$
(20)

The relaxation time, T_r , can be derived similarly as in one dimension:

$$T_r = \frac{L_r}{|a_n| + \frac{\nu}{L_r}},\tag{21}$$

with a projected advection speed, $a_n = an_x + bn_y$, for an arbitrary vector, $\mathbf{n} = (n_x, n_y)$, in relation to the projected Jacobian:

$$\mathbf{A}_{n} = \mathbf{A}n_{x} + \mathbf{B}n_{y} = \begin{bmatrix} a_{n} & -\nu n_{x} & -\nu n_{y} \\ -n_{x}/T_{r} & 0 & 0 \\ -n_{y}/T_{r} & 0 & 0 \end{bmatrix}.$$
 (22)

 $4 \ {\rm of} \ 24$

American Institute of Aeronautics and Astronautics

The eigenvalues are given by

$$\lambda_1 = a_n^- \left(1 - \frac{1}{Re_{L_r}^-} \right), \quad \lambda_2 = a_n^+ \left(1 + \frac{1}{Re_{L_r}^+} \right), \quad \lambda_3 = 0,$$
(23)

where $a_n^- = \min(0, a_n)$ and $a_n^+ = \max(0, a_n)$. The vanishing eigenvalue, λ_3 , is associated with the inconsistency damping mode: it purely damps out any inconsistency (nonzero $q_x - p_y$) that may be contained in an initial solution but must vanish at a steady state[1].

On the boundary, since u is given, we can specify the gradient variables in the direction along the boundary: p at y = 0 or y = 1, and q at x = 0 or x = 1. Elsewhere, the gradient variables (the normal gradients) will be computed by a numerical scheme. For problems with the Neumann boundary condition, the gradient variables can be directly specified on the boundary. In effect, the Neumann condition turns into the Dirichlet condition in the first-order system approach.

IV. Discretization of 1D First-Order Advection-Diffusion System

We construct a numerical scheme for solving the first-order advection-diffusion system (5) based on, for the sake of convenience, a residual-distribution approach: compute a cell-residual and distribute it to the nodes. The resulting scheme can be viewed, as will be shown later, as a finite-volume or finite-difference scheme. In any case, it will be a compact three-point difference scheme, involving only the nearest neighbors at every data point.

We begin by generating a set of nodes, $\{J\}$, with coordinates, x_j , distributed arbitrary over the domain of interest. With the solutions stored at each node, (u_j, p_j) , $j \in \{J\}$, and two boundary conditions given for u, the task is to compute the steady state solution, $\{u_j\}$, at the interior nodes and $\{p_j\}$ at all nodes. Note, as in the pure diffusion case[1] that the number of unknowns will be exactly equal to the number of cell-residuals: all the cell-residuals can be driven to zero exactly at a steady state, thus implying the existence of a unique solution set for linear problems. We remark that this is not true for scalar residual-distribution schemes (and any cell-vertex type schemes) that directly solve (4), resulting in a discrete problem overdetermined by one extra cell-residual.

To begin residual-distribution, we first define the cell-residual, Φ^C , as an integral value of the equations over the cell, $C = [x_j, x_{j+1}]$,

$$\Phi^C = \begin{bmatrix} \Phi_1^C \\ \Phi_2^C \end{bmatrix} = \int_{x_j}^{x_{j+1}} (-\mathbf{A}\mathbf{U}_x + \mathbf{Q}) \, dx \tag{24}$$

$$= \begin{bmatrix} -a(u_{j+1} - u_j) + \nu(p_{j+1} - p_j) \\ (u_{j+1} - u_j)/T_r - (w_{j+1}^C p_{j+1} + w_j^C p_j)h^C/T_r \end{bmatrix}$$
(25)

where $h^C = x_{j+1} - x_j$, and (w_j^C, w_{j+1}^C) is a set of quadrature weights that satisfy, within the cell,

$$w_j^C + w_{j+1}^C = 1. (26)$$

The choice of the weights is left open at this point; it will be discussed in the next subsection. Second, we distribute the cell-residual to the nodes, j and j + 1, by using the upwind distribution matrix:

$$\mathcal{B}_{j}^{C} = \frac{1}{2} \mathbf{R} \begin{bmatrix} \left(1 - \frac{\lambda_{1}}{|\lambda_{1}|}\right) & 0\\ 0 & \left(1 - \frac{\lambda_{2}}{|\lambda_{2}|}\right) \end{bmatrix} \mathbf{R}^{-1} = \frac{1}{Re_{L_{r}} + 2} \begin{bmatrix} 1 & L_{r}\\ \frac{Re_{L_{r}} + 1}{L_{r}} & 1 + Re_{L_{r}} \end{bmatrix}, \quad (27)$$

$$\mathcal{B}_{j+1}^{C} = \frac{1}{2} \mathbf{R} \begin{bmatrix} \left(1 + \frac{\lambda_{1}}{|\lambda_{1}|}\right) & 0\\ 0 & \left(1 + \frac{\lambda_{2}}{|\lambda_{2}|}\right) \end{bmatrix} \mathbf{R}^{-1} = \frac{1}{Re_{L_{r}} + 2} \begin{bmatrix} 1 + Re_{L_{r}} & -L_{r}\\ -\frac{Re_{L_{r}} + 1}{L_{r}} & 1 \end{bmatrix}, \quad (28)$$

American Institute of Aeronautics and Astronautics



Figure 2. Distribution of cell-residuals in one dimension.

which projects the residual onto characteristic subspaces, distributes the projected residuals to the left or right according to the sign of the characteristic speed. After completing the distribution step within all elements, we arrive at the following semi-discrete equation,

$$\widetilde{h}_j \frac{d\mathbf{U}_j}{dt} = \left[\mathcal{B}_j^L \Phi^L + \mathcal{B}_j^R \Phi^R \right],$$
(29)

where L and R denote the cells on the left and right of the node j respectively, and \tilde{h}_j is the measure of the dual control volume around the node, j, defined by

$$\widetilde{h}_j = \frac{h^L + h^R}{2}.$$
(30)

(See Figure 2.) Finally, we integrate the semi-discrete equation in time towards a steady state. In this study, we employ the forward Euler time stepping.

$$\widetilde{h}_{j} \frac{\mathbf{U}_{j}^{n+1} - \mathbf{U}_{j}^{n}}{\Delta t} = \left[\mathcal{B}_{j}^{L} \Phi^{L} + \mathcal{B}_{j}^{R} \Phi^{R} \right],$$
(31)

where the right hand side is evaluated at the time level n. This is the fully-discrete *explicit* upwind residualdistribution scheme for the first-order advection-diffusion system. Note that this scheme is a compact *three-point difference scheme* involving only the two neighbor nodes for all Reynolds numbers.

To see how advection and diffusion schemes are combined into one, note first that the distribution matrices (28) are nothing but the projection matrices:

$$\mathcal{B}_j^C = \mathbf{\Pi}_1, \quad \mathcal{B}_{j+1}^C = \mathbf{\Pi}_2, \tag{32}$$

and recall that each projection matrix can be written as a linear combination of its advection and diffusion limits, i.e., (16). Then, we can express the distribution matrices as

$$\mathcal{B}_{j}^{C} = \frac{Re_{L_{r}}}{Re_{L_{r}} + 2} \begin{bmatrix} 0 & 0\\ \\ \frac{1}{L_{r}} & 1 \end{bmatrix} + \frac{2}{Re_{L_{r}} + 2} \begin{bmatrix} \frac{1}{2} & \frac{L_{r}}{2}\\ \\ \frac{1}{2L_{r}} & \frac{1}{2} \end{bmatrix},$$
(33)

$$\mathcal{B}_{j+1}^{C} = \frac{Re_{L_{r}}}{Re_{L_{r}}+2} \begin{bmatrix} 1 & 0\\ \\ -\frac{1}{L_{r}} & 0 \end{bmatrix} + \frac{2}{Re_{L_{r}}+2} \begin{bmatrix} \frac{1}{2} & \frac{-L_{r}}{2}\\ \\ -\frac{1}{2L_{r}} & \frac{1}{2} \end{bmatrix}.$$
(34)

This shows that the upwind distribution matrix (28) can be viewed as a weighted average of upwind distribution matrices of advection and diffusion.

IV.A. Source Term Discretization and Cell Reynolds Number

Note that the upwind distribution does not guarantee monotone solutions at a steady state. To see this, suppose we employ the trapezoidal rule to discretize the source term in (25):

$$w_j^C = w_{j+1}^C = \frac{1}{2},\tag{35}$$

which ensures second-order accuracy of the cell-residual, Φ^C , and recall that all cell-residuals vanish at a steady state, i.e.,

$$\Phi^L = \Phi^R = \mathbf{0},\tag{36}$$

for the left and right cells of all $j \in \{J\}$. Then, we find from this pair of vanishing cell-residuals with $h = h^L = h^R$ that the steady state solutions, both u_j and p_j , satisfy

$$a \frac{u_{j+1} - u_{j-1}}{2h} = \nu \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2},$$
(37)

i.e., we have

$$u_{j} = \frac{1}{2} \left(1 - \frac{Re_{h}}{2} \right) u_{j+1} + \frac{1}{2} \left(1 + \frac{Re_{h}}{2} \right) u_{j-1}, \tag{38}$$

where Re_h is the cell Reynolds number defined by

$$Re_h \equiv \frac{ah}{\nu}.\tag{39}$$

This corresponds to the classical central-difference approximation to the steady advection-diffusion equation; it is prone to spurious oscillations because the coefficient for u_{j+1} goes negative when $Re_h > 2$. Note that this is derived from the cell-residuals only, and has nothing to do with the distribution matrix. To avoid oscillations, therefore, we must modify the cell-residuals, such that they correspond to an upwind discretization at the steady state. This is possible through the source term, and in fact, a one-sided evaluation of the source term,

$$w_j^C = 1, \quad w_{j+1}^C = 0, \quad C = L \text{ or } R,$$
(40)

leads to the desired upwind discretization of the steady equation:

$$a \frac{u_j - u_{j-1}}{h} = \nu \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2},\tag{41}$$

i.e.,

$$u_j = \frac{Re_h + 1}{Re_h + 2} u_{j+1} + \frac{1}{Re_h + 2} u_{j-1}, \tag{42}$$

and the same for p_j . Observe that all coefficients are now positive for all Re_h . The accuracy of the cell-residual, however, deteriorates to first-order for this choice.

IV.B. O(h) Time Step

Since the first-order advection-diffusion system is hyperbolic, the time step for our explicit scheme is restricted by the CFL condition:

$$\Delta t = \text{CFL} \frac{h_{min}}{a + \nu/L_r},\tag{43}$$

where CFL is the CFL number (≤ 1), h_{min} is the minimum mesh size, and $a + \nu/L_r$ is the maximum wave speed. Since we have $L_r = O(1)$, the time step is proportional to the mesh size, i.e., $\Delta t = O(h)$, for all Reynolds numbers, even in the diffusion limit ($a \rightarrow 0$). Compare this with the well-known time step restriction for a common scalar scheme, i.e., (37) or (41) augmented with the forward Euler time-stepping:

$$\Delta t = \text{CFL} \frac{h_{min}}{a + 2\nu/h_{min}}.$$
(44)

This is generally $O(h^2)$ unless advection dominates. We emphasize that O(h) time step (43) is a direct consequence of solving the *hyperbolic* advection-diffusion system, not a property of the particular numerical

scheme. Any explicit schemes, e.g., finite-volume or finite-element schemes, developed for the hyperbolic advection-diffusion system will allow this remarkably large time step.

To see an impact of the size of time steps on the number of iterations towards a steady state, suppose that a steady state is reached physically at $t = t_{steady}$ and numerically at n_f iterations. Then, we have

$$t_{steady} = n_f \,\Delta t. \tag{45}$$

Since t_{steady} depends not on numerical schemes, we take it as a constant for a given problem and therefore we have

$$n_f \propto \frac{1}{\Delta t}.$$
 (46)

Now, since $h \propto 1/N$ where N is the number of nodes, we obtain

r

$$n_f = \begin{cases} O(N) & \text{for } \Delta t = O(h), \\ O(N^2) & \text{for } \Delta t = O(h^2). \end{cases}$$
(47)

This shows that the number of iterations is proportional to the number of unknowns with O(h) time step. This is a tremendous advantage. In the previous paper[1], we showed for pure diffusion problems that a scheme for the hyperbolic diffusion system was indeed orders of magnitude faster than the Jacobi iteration which corresponds to a method with $O(h^2)$ time step. Note that we have O(h) time step not just in one dimension but in two and three dimensions simply because the system is hyperbolic for all dimensions and thus the CFL condition (43) applies. Therefore, we have, in two dimensions where $h \propto 1/N^{\frac{1}{2}}$,

$$h_f = \begin{cases} O(N^{\frac{1}{2}}) & \text{for } \Delta t = O(h), \\ O(N) & \text{for } \Delta t = O(h^2), \end{cases}$$
(48)

and in three dimensions where $h \propto 1/N^{\frac{1}{3}}$,

$$n_f = \begin{cases} O(N^{\frac{1}{3}}) & \text{for } \Delta t = O(h), \\ O(N^{\frac{2}{3}}) & \text{for } \Delta t = O(h^2). \end{cases}$$
(49)

In all dimensions, therefore, the ratio of n_f is proportional to the number of unknowns, meaning that the gain will grow substantially as the number of unknowns increases. We point out that this type of convergence, i.e., the one with $\Delta t = O(h)$, is not observed in general by stationary iterative methods such as the Jacobi, the Gauss-Seidel, or the successive over-relaxation (which all correspond to $O(h^2)$ time step) for diffusion dominated problems.

For high Reynolds number applications, the minimum mesh size can be extremely small so that an explicit time-stepping with O(h) time step (although significantly larger than $O(h^2)$) may still limit its practical value. We may still need to resort to some implicit treatment. Even if that is the case, the present approach may offer advantages over common schemes: simpler Jacobian matrix formation with compact stencils and/or fast iterative solvers (comparable to Krylov-type methods), etc. These possibilities will be investigated in future.

IV.C. Accuracy

Expand smooth functions u and p around node j, and substitute them into the semi-discrete equation (29) to get

$$\frac{d\mathbf{U}_{j}}{dt} = \frac{1}{\widetilde{h}_{j}} \left(\mathcal{B}_{j}^{R} h^{R} + \mathcal{B}_{j}^{L} h^{L} \right) \left(-\mathbf{A}\mathbf{U}_{x} + \mathbf{Q} \right) + \frac{1}{2\widetilde{h}_{j}} \left\{ \mathcal{B}_{j}^{R} (h^{R})^{2} - \mathcal{B}_{j}^{L} (h^{L})^{2} \right\} \left(-\mathbf{A}\mathbf{U}_{x} + \mathbf{Q} \right)_{x}
+ \frac{1}{\widetilde{h}_{j}} \left[\mathcal{B}_{j}^{R} \left(w_{j}^{R} - \frac{1}{2} \right) (h^{R})^{2} - \mathcal{B}_{j}^{L} \left(w_{j}^{L} - \frac{1}{2} \right) (h^{L})^{2} \right] \mathbf{Q}_{x} + O(h^{2}).$$
(50)

American Institute of Aeronautics and Astronautics

If these functions are the exact steady solutions of the first-order system, the time derivative as well as the spatial terms on the right vanish, and we are left with the terms on the second line, i.e., the local truncation error, $T\mathcal{E}$,

$$\mathcal{TE} = \frac{1}{\widetilde{h}_j} \left[\mathcal{B}_j^R \left(w_j^R - \frac{1}{2} \right) (h^R)^2 - \mathcal{B}_j^L \left(w_j^L - \frac{1}{2} \right) (h^L)^2 \right] \mathbf{Q}_x + O(h^2).$$
(51)

If the trapezoidal rule is used to discretize the source term on both cells, the leading term will vanish and the second-order accuracy is obtained at the steady state. But if the one-sided quadrature is used in either cell, it remains finite and the accuracy reduces to first-order.

IV.D. Finite-Volume Form

Here, we show that our scheme is a finite-volume scheme. First, we express the products of the distribution matrix and \mathbf{A} as

$$\mathcal{B}_{j}^{C}\mathbf{A} = \mathbf{\Pi}_{1}\mathbf{A} = \mathbf{\Pi}_{1}\left(\lambda_{1}\mathbf{\Pi}_{1} + \lambda_{2}\mathbf{\Pi}_{2}\right) = \lambda_{1}\mathbf{\Pi}_{1} = \frac{1}{2}\left(\mathbf{A} - |\mathbf{A}|\right),$$
(52)

$$\mathcal{B}_{j+1}^{C}\mathbf{A} = \mathbf{\Pi}_{2}\mathbf{A} = \mathbf{\Pi}_{2}\left(\lambda_{1}\mathbf{\Pi}_{1} + \lambda_{2}\mathbf{\Pi}_{2}\right) = \lambda_{2}\mathbf{\Pi}_{2} = \frac{1}{2}\left(\mathbf{A} + |\mathbf{A}|\right),\tag{53}$$

Then, we expand and rewrite the semi-discrete scheme (29) as

$$\widetilde{h}_j \frac{d\mathbf{U}_j}{dt} = \mathcal{B}_j^L \Phi^L + \mathcal{B}_j^R \Phi^R$$
(54)

$$= -\frac{1}{2} \left(\mathbf{A} + |\mathbf{A}| \right) \Delta \mathbf{U}^{L} + \mathbf{\Pi}_{2} \overline{\mathbf{Q}}^{L} h^{L} - \frac{1}{2} \left(\mathbf{A} - |\mathbf{A}| \right) \Delta \mathbf{U}^{R} + \mathbf{\Pi}_{1} \overline{\mathbf{Q}}^{R} h^{R}$$
(55)

$$= -\frac{1}{2} \left(\mathbf{f}_{j+1} - \mathbf{f}_j - |\mathbf{A}| \Delta \mathbf{U}^R \right) + \mathbf{\Pi}_1 \overline{\mathbf{Q}}^R h^R + \frac{1}{2} \left(-\mathbf{f}_j + \mathbf{f}_{j-1} - |\mathbf{A}| \Delta \mathbf{U}^L \right) + \mathbf{\Pi}_2 \overline{\mathbf{Q}}^L h^L \quad (56)$$

$$= -\frac{1}{2} \left(\mathbf{f}_j + \mathbf{f}_{j+1} - |\mathbf{A}| \Delta \mathbf{U}^R \right) + \frac{1}{2} \left(\mathbf{f}_j + \mathbf{f}_{j-1} - |\mathbf{A}| \Delta \mathbf{U}^L \right) + \mathbf{\Pi}_2 \overline{\mathbf{Q}}^L h^L + \mathbf{\Pi}_1 \overline{\mathbf{Q}}^R h^R, \quad (57)$$

where

$$\mathbf{f}_{j} = \mathbf{A}\mathbf{U}_{j} = \begin{bmatrix} au_{j} - \nu p_{j} \\ -u_{j}/T_{r} \end{bmatrix}.$$
(58)

Therefore, our scheme is a finite-volume scheme:

$$\widetilde{h}_{j}\frac{d\mathbf{U}_{j}}{dt} = -\left[\mathbf{F}_{j+1/2} - \mathbf{F}_{j-1/2}\right] + \widetilde{\mathbf{Q}}_{j},$$
(59)

where

$$\mathbf{F}_{j+1/2} = \frac{1}{2} \left(\mathbf{f}_j + \mathbf{f}_{j+1} - |\mathbf{A}| \Delta \mathbf{U}^R \right), \tag{60}$$

$$\mathbf{F}_{j-1/2} = \frac{1}{2} \left(\mathbf{f}_j + \mathbf{f}_{j-1} - |\mathbf{A}| \, \Delta \mathbf{U}^L \right), \tag{61}$$

$$\widetilde{\mathbf{Q}}_{j} = \mathbf{\Pi}_{1} \overline{\mathbf{Q}}^{R} h^{R} + \mathbf{\Pi}_{2} \overline{\mathbf{Q}}^{L} h^{L}.$$
(62)

This apparently first-order upwind finite-volume scheme is second-order accurate at a steady state provided the trapezoidal rule is used within each cell for $\overline{\mathbf{Q}}^L$ and $\overline{\mathbf{Q}}^R$. In effect, the particular source term discretization (62) makes it possible to achieve second-order accuracy at a steady state without reconstructing the solution at the cell interface.

V. Discretization of 2D First-Order Advection-Diffusion System

In two dimensions, we present a finite-difference scheme for structured grids and a multidimensional upwind scheme for unstructured triangular grids. We emphasize that these are just examples and various other discretization methods can also be applied. Many properties of the resulting schemes come from solving the first-order advection-diffusion system (18), not specific to a particular numerical scheme.



Figure 3. Structured grid of non-uniform spacing.

V.A. Structured Grids

We consider a structured grid with non-uniform spacing in the square domain Ω as in Figure 3, with N_x points distributed along each line in x-direction and N_y points in y-direction. To construct a numerical scheme for computing the solution at each grid point, $(x_{i,j}, y_{i,j})$, we first decompose the advection-diffusion system (18) into two: the one in x-direction

$$u_t + au_x = \nu p_x,$$

 $p_t = (u_x - p)/T_r^x,$
(63)

and the other in y-direction,

$$u_t + bu_y = \nu q_y,$$

$$q_t = (u_y - q)/T_r^y,$$
(64)

where T_r^x and T_r^y are defined in the same way as in one dimension with the advection speeds, a and b, respectively:

$$T_{r}^{x} = \frac{L_{r}^{x}}{a\left(1 + \frac{1}{|Re_{L_{r}^{x}}|}\right)} \quad , \quad Re_{L_{r}^{x}} = \frac{aL_{r}^{x}}{\nu}, \tag{65}$$

$$T_r^y = \frac{L_r^y}{b\left(1 + \frac{1}{|Re_L^y|}\right)} \quad , \quad Re_{L_r^y} = \frac{bL_r^y}{\nu} \tag{66}$$

and the length scales, L_r^x and L_r^y are defined by the optimal formula (8) with a and b, respectively. Each problem now being one dimensional, we simply apply the one-dimensional scheme in each direction. In the

10 of 24



Figure 4. Stencil for a structured grid scheme.

form of one-dimensional residual-distribution, the resulting scheme can be written as

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = \operatorname{\mathbf{Res}}_{i,j}^{x}(1) + \operatorname{\mathbf{Res}}_{i,j}^{y}(1), \tag{67}$$

$$\frac{p_{i,j}^{n+1} - p_{i,j}^n}{\Delta t} = \mathbf{Res}_{i,j}^x(2), \tag{68}$$

$$\frac{q_{i,j}^{n+1} - q_{i,j}^n}{\Delta t} = \mathbf{Res}_{i,j}^y(2), \tag{69}$$

where $\operatorname{Res}_{i,j}^{x}$ and $\operatorname{Res}_{i,j}^{y}$ are residual vectors evaluated at time level *n* by the one-dimensional scheme applied for (63) and (64) respectively. The time step is defined by

$$\Delta t = \min\left(\frac{\text{CFL}\Delta x_{min}}{a + \nu/L_r^x}, \frac{\text{CFL}\Delta y_{min}}{b + \nu/L_r^y}\right),\tag{70}$$

where $\text{CFL} \leq 1$. Note that the trapezoidal rule has been used in the source term discretization here; a one-sided quadrature may be used to avoid oscillations when the cell Reynolds number is greater than 2. On the boundary, where the gradient variables need to be computed, we simply ignore Φ^L or Φ^R , whichever does not exist. The scheme is a compact five-point difference scheme (see Figure 4), and it is second-order accurate at a steady state. This scheme can be extended to three dimensions straightforwardly.

It is illustrative to consider the pure diffusion limit of the scheme: (a, b) = (0, 0) and $\nu = 1$. This means that the scheme becomes an iterative Laplace solver. Since the solution is expected to be smooth, it is reasonable to assume a uniform mesh with a spacing, h, in both x and y directions. Then, the scheme can be written as

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = \left(1 - \frac{h}{2L_{r}}\right) \left(\frac{p_{i+1,j}^{n} - p_{i-1,j}^{n}}{2h} + \frac{q_{i,j+1}^{n} - q_{i,j-1}^{n}}{2h}\right) + \frac{h}{2L_{r}} \frac{u_{i-1,j}^{n} + u_{i+1,j}^{n} + u_{i,j-1}^{n} + u_{i,j+1}^{n} - 4u_{i,j}^{n}}{h^{2}},$$
(71)

$$\frac{p_{i,j}^{n+1} - p_{i,j}^n}{\Delta t} = \frac{1}{T_r} \left(\frac{u_{i+1,j}^n - u_{i-1,j}^n}{2h} - p_{i,j}^n \right) + \frac{h}{2L_r} \left(1 - \frac{h}{2L_r} \right) \frac{p_{i+1,j}^n - 2p_{i,j}^n + p_{i-1,j}^n}{h^2}, \tag{72}$$

$$\frac{q_{i,j}^{n+1} - q_{i,j}^n}{\Delta t} = \frac{1}{T_r} \left(\frac{u_{i,j+1}^n - u_{i,j-1}^n}{2h} - q_{i,j}^n \right) + \frac{h}{2L_r} \left(1 - \frac{h}{2L_r} \right) \frac{q_{i,j+1}^n - 2q_{i,j}^n + q_{i,j-1}^n}{h^2}, \tag{73}$$

where

$$\Delta t = \operatorname{CFL} hL_r,\tag{74}$$

and

$$L_r = L_r^x = L_r^y = \frac{1}{\sqrt{2}\pi}, \quad T_r = T_r^x = T_r^y = L_r^2 = \frac{1}{2\pi^2}.$$
 (75)

$11~{\rm of}~24$

American Institute of Aeronautics and Astronautics



Figure 5. Distribution of a non-zero cell-residual to the set of vertices $\{i_T\} = \{1, 2, 3\}$.



Figure 6. Median dual cell around node j in the set of triangles sharing that node $\{T_j\}$.

This is a two-dimensional Laplace solver (a Jacobi iteration scheme) which converges at $O(N^{\frac{1}{2}})$ iterations which is comparable to Krylov-type methods. It is interesting to note that if we set $L_r = h/2$ instead, then the main variable, $u_{i,j}$, decouples from the gradient variables, $p_{i,j}$ and $q_{i,j}$: the scheme becomes the Jacobi iteration scheme for the main variable and the gradient variables can be explicitly computed from it. But in this case we have $\Delta t = O(h^2)$, implying O(N) iterations.

V.B. Unstructured Triangular Grids

For unstructured triangular grids, we begin by dividing the domain into a set of triangles $\{T\}$ and a set of nodes $\{V\}$, and store the solution at each node $(u_j, p_j), j \in \{V\}$. Now, the task is to compute the steady state solution $\{u_j\}$ at the interior nodes, and $\{p_j, q_j\}$ at all nodes except for the boundary nodes on which they can be computed from u on the boundary.

To discretize the first-order advection-diffusion system on the triangular grid, we employ the residualdistribution method. We first define the cell-residual over cell T (see Figure 5),

$$\Phi^T = \int_T \left(-\mathbf{A}\mathbf{U}_x - \mathbf{B}\mathbf{U}_y + \mathbf{Q} \right) \, dx dy. \tag{76}$$

Assuming a piecewise linear variation of \mathbf{U} over the cell, we obtain

$$\Phi^T = -\sum_{i=1}^3 \mathbf{K}_i \mathbf{U}_i + \overline{\mathbf{Q}}_T S_T,$$
(77)

where

$$\mathbf{K}_{i} = \frac{1}{2} \left(\mathbf{A}, \mathbf{B} \right) \cdot \mathbf{n}_{i}, \ \overline{\mathbf{Q}}_{T} = \frac{\mathbf{Q}_{1} + \mathbf{Q}_{2} + \mathbf{Q}_{3}}{3},$$
(78)

 $\mathbf{n}_i = (n_{i_x}, n_{i_y})$ is the inward scaled normal (see Figure 6), and the source term has been discretized to ensure the exactness for linear functions. For the derivatives, \mathbf{U}_x and \mathbf{U}_y , this corresponds precisely to the Green-Gauss integration. We remark here that in the definition of the cell-residual above, we set

$$T_r = \frac{L_r}{|\mathbf{a}| + \frac{\nu}{L_r}},\tag{79}$$

where $|\mathbf{a}| = \sqrt{a^2 + b^2}$, so that T_r is constant within the cell. This is to ensure the residual property on the equations, $u_x - p$ and $u_y - q$: no updates will be sent to the nodal solutions if these equations are satisfied

$12 \ {\rm of} \ 24$

exactly over the cell in the integral sense. We now distribute the cell-residual to the nodes. In this work, we employ the matrix LDA scheme[11,12], which is an upwind scheme defined by

$$\mathcal{B}_{i}^{T} = \mathbf{K}_{i}^{+} \left(\sum_{i=1,2,3} \mathbf{K}_{i}^{+} \right)^{-1}, \quad \mathbf{K}_{i}^{+} = \mathbf{R}_{n_{i}} \mathbf{\Lambda}_{n_{i}}^{+} \mathbf{R}_{n_{i}}^{-1}, \tag{80}$$

where we do not use (79) in computing the distribution matrix to properly account for the characteristics of the hyperbolic system. The LDA scheme generally gives second-order accurate solutions at a steady state. This is due to the residual property: the cell-residual vanishes for linear exact solutions[13]. After performing the distribution step all over the cells, we obtain the following semi-discrete equation at each node:

$$\frac{d\mathbf{U}_j}{dt} = \frac{1}{S_j} \sum_{T \in \{T_j\}} \mathcal{B}_j^T \Phi^T, \tag{81}$$

where S_j is the medial dual cell area (see Figure 6). This is then integrated in time by the forward Euler time stepping to reach a steady state.

Note that the Green-Gauss evaluation of the solution derivatives in the cell-residuals for the equations for p and q, Φ_2^T and Φ_3^T , are exact only for linear u. This means that although the source term has been designed to be exact for linear gradients, the whole residuals can be exact only for linear solution, u, and thus only for constant gradients, p and q. Therefore, first-order accuracy is expected for the gradients in general. For smooth grids, it recovers second-order accuracy in the nodal residual[1], but for irregular grids, it is indeed first-order accurate for the gradients as demonstrated in [14]. To achieve second-order accuracy for the gradient variables on arbitrary grids, we need to improve the accuracy of the Green-Gauss term such that it will be exact for quadratic solution, u. This can be easily achieved in our case, since the gradients are available, by adding a high-order curvature correction to the Green-Gauss term[14, 15]:

$$\Phi_2^T = \frac{1}{T_r} \int_T (u_x - p) \, dx dy = \frac{1}{T_r} \left(\frac{1}{2} \sum_{i=1}^3 (u_i + \delta_i) n_{ix} - \overline{p}_T S_T \right), \tag{82}$$

$$\Phi_3^T = \frac{1}{T_r} \int_T (u_y - q) \, dx dy = \frac{1}{T_r} \left(\frac{1}{2} \sum_{i=1}^3 (u_i + \delta_i) n_{iy} - \overline{q}_T S_T \right), \tag{83}$$

where δ_i is the high-order correction term given by

$$\delta_i = -\frac{1}{6} \left(\Delta p_i \Delta x_i + \Delta q_i \Delta y_i \right), \tag{84}$$

where Δp_i denotes the difference of the nodal values of p taken counterclockwise along the edge opposite to the node i (e.g., $\Delta p_1 = p_2 - p_3$), and similarly for others. This corresponds to using Simpson's rule along each edge with a midpoint value reconstructed by the Hermite interpolation (see [14]). These 'corrected' residuals are now exact for quadratic solution, u, and thus exact for linear gradients, p and q, ensuring the second-order accuracy of the gradient variables. Note that the correction term does not require any explicit gradient reconstruction (which was required in [14, 15]) since we now carry the gradients as unknowns and they are directly available at nodes.

We remark that it is possible to make the same high-order correction also to the advective term to devise a third-order scheme for u. However, we do not consider such a scheme here because it is a rather special case for a simple scalar equation and does not extend in general to systems of equations. We also point out that we expect third-order accuracy in the solution, u, in the diffusion limit. This is because the cell-residual for the advection-diffusion equation will be dominated by the diffusion term which is exact for linear gradients, meaning exact for quadratic solution.

The source term quadrature in (78) has been chosen to ensure second-order accuracy above, but it is likely to produce oscillatory solutions for high Reynolds number cases. An upwind quadrature is required for monotonicity. One possible formula is the following:

$$\overline{p}_T = w_1^T p_1 + w_2^T p_2 + w_3^T p_3, \tag{85}$$

$$\overline{q}_T = w_1^T q_1 + w_2^T q_2 + w_3^T q_3, \tag{86}$$

$13 {\rm ~of~} 24$



Figure 7. Quadrature weights given by (87) in the case of two upwind nodes. In actual implementation, we set b = 0 for the integration of p while a = 0 for the integration of q.

where

$$w_i^T = \frac{k_i^-}{\sum_{m=1}^3 k_m^-}, \quad i = 1, 2, 3,$$
(87)

$$k_i^- = \min(0, k_i), \quad k_i = \frac{1}{2}(a, b) \cdot \mathbf{n}_i,$$
(88)

and we set b = 0 for \overline{p}_T while a = 0 for \overline{q}_T . It is easy to see that we have $k_i^- = 1$ if the node *i* is the only upwind node. If there are two upwind nodes, k_i^- gives a fraction of the triangle defined by the other two nodes and the intersection point of the line along (a, b) passing through the downwind node to the area of the triangle, T (see Figure 7). In either case, the quadrature weights for the nodes in the downwind side will be effectively set to be zero. This gives monotone solutions, but the accuracy reduces to first-order.

VI. Results

VI.A. One-Dimensional Problem

We consider the following problem:

$$u_t + au_x = \nu \, u_{xx} + q(x) \quad \text{in } \Omega = (0, 1),$$
(89)

with u(0) = 0 and u(1) = 1, where

$$q(x) = \frac{\pi}{Re} \left[a\cos(\pi x) + \pi\nu\sin(\pi x) \right],\tag{90}$$

where $Re = a/\nu$. The source term has been introduced to make the steady state solution non-trivial in the diffusion limit. The exact steady state solution is given by

$$u_{exact}(x) = \frac{\exp(-Re) - \exp(xRe - Re)}{\exp(-Re) - 1} + \frac{1}{Re}\sin(\pi x).$$
(91)

This is a smooth sine curve in the diffusion limit, but develops a narrow boundary layer near x = 1 when advection dominates (see Figure 8).

We compute the steady state solution to this problem, by solving the equivalent first-order system:

$$u_t + au_x = \nu p_x + q(x),$$

 $p_t = (u_x - p)/T_r,$
(92)

$$14 \text{ of } 24$$



Figure 8. Exact solutions (solid curves) and numerical solutions obtained by the upwind advection-diffusion scheme (symbols: triangles, circles, squares) for Re = 1, 10, 100. A computational grid (33 nodes) used in the numerical experiment is shown by stars in the bottom.

with the upwind advection-diffusion scheme developed in Subsection IV. The source term in the first equation does not affect the advection discretization (unlike the one in the second equation as discussed in Subsection IV.A), and therefore it is evaluated by the trapezoidal rule over each cell to ensure the second-order accuracy and added to the cell-residual. We start from the initial solution, $(u, p) = (x^2, 2x)$, and integrate in time until convergence by the forward Euler method. The method is taken to be converged when the nodal residuals are reduced five orders of magnitude in the L_1 norm. The time step is taken global with CFL = 0.99. We conducted numerical experiments with non-uniform grids with nodes, N = 33, 65, 129, 257. Each grid was generated from a uniform grid by the following mapping:

$$x_i = \frac{1 - \exp(-\alpha\xi_i)}{1 - \exp(-\alpha)},\tag{93}$$

where $\xi_i = (i-1)/(N-1)$, i = 1, 2, 3, ..., N, and $\alpha = 4.5$ for all nodes and grids (see Figure 8 for an example). Results were obtained for a wide range of the Reynolds numbers: $Re = 10^k$, where k = -3, -2, -1.5, -1, -0.5, 0, 0.5, 1, 1.5, 2, 3.

Table 1 shows the iteration numbers obtained. Remarkably, the number of iterations to reduce the residuals five orders of magnitude is nearly *independent of the Reynolds number*. This is considered due to O(h) time step for all Reynolds numbers and also to the perfect preconditioning of the first-order system by the optimal L_r . We point out that the convergence in the diffusion limit can be much faster in practice because grid stretching is not required and a uniform grid can be safely employed (a much larger minimum mesh spacing than stretched grids).

To demonstrate the impact of O(h) time step on the number of iterations, we compute the same steady state solution by integrating the scalar advection-diffusion equation in time, again until the residual is reduced by five orders of magnitude, by a spatially second-order Galerkin scheme derived with a continuous piecewise linear basis function over a non-uniform grid:

$$u_{j}^{n+1} = u_{j}^{n} + \frac{2\Delta t}{x_{j+1} - x_{j-1}} \left[-a \frac{u_{j+1}^{n} - u_{j-1}^{n}}{2} + \nu \left(\frac{u_{j+1}^{n} - u_{j}^{n}}{x_{j+1} - x_{j}} - \frac{u_{j}^{n} - u_{j-1}^{n}}{x_{j} - x_{j-1}} \right) \right],$$
(94)

 $15~{\rm of}~24$

logio Re	Number of Iterations			
10810 100	33 nodes	65 nodes	129 nodes	257 nodes
-3.0	2976	7368	14685	29170
-2.0	2979	7376	14700	29199
-1.5	2986	7393	14735	29270
-1.0	3010	7449	14847	29497
-0.5	3086	7629	15218	30244
0.0	3349	8186	16491	32869
0.5	3175	7926	17277	38081
1.0	3999	7735	15428	35747
1.5	3062	7180	15389	32277
2.0	3214	6458	13962	29518
3.0	3286	6877	14355	29893

Table 1. The number of iterations for the advection-diffusion scheme to reduce the nodal-residuals by five orders of magnitude.

where the time step is restricted by (44), with CFL = 0.99, which is $O(h^2)$ in general. Figure 9 shows the iteration number versus the mesh size, for the optimal L_r and for the scalar Galerkin scheme. It is clearly seen that the number of iterations for the scalar scheme increases quadratically with the mesh size even for fairly advection-dominated cases. On the other hand, for our scheme, the number of iterations grows linearly with the mesh size for all Reynolds numbers. This is a natural consequence of solving the system that is hyperbolic for all Reynolds numbers.

Figures 10(a) and 10(b) show the L_{∞} error convergence results: 10(a) for the main variable, u, 10(b) for the gradient variable, p, respectively. Here, for a better visibility, we shifted the results with respect to those for $Re = 10^{-3}$ so that it can be read from the top to the bottom for increasing Reynolds numbers (only the errors for $Re = 10^{-3}$ can be correctly read off from the numbers on the vertical axis). The figure clearly shows that our upwind advection-diffusion scheme is uniformly second-order accurate for all Reynolds numbers and all variables, including the gradient variable on the boundary. This demonstrates that the scheme maintain its accuracy through the boundary.

In this numerical experiment, we employed the trapezoidal rule in the source term discretization for all cases. Although the condition, $Re_h \leq 2$, is violated in some region, since it is almost always satisfied near the narrow layer, we do not observe any serious oscillations (see Figure 8).

VI.B. Two-Dimensional Problems

We now consider the two-dimensional advection-diffusion problem:

 u_t

$$u_t + au_x + bu_y = \nu \ (u_{xx} + u_{yy}) \quad \text{in } \Omega = (0, 1) \times (0, 1), \tag{95}$$

with the solution specified on the boundary by the following boundary-layer type exact steady solution,

$$u(x,y) = \frac{\left[1 - \exp\left((x-1)\frac{a}{\nu}\right)\right] \left[1 - \exp\left((y-1)\frac{b}{\nu}\right)\right]}{\left[1 - \exp\left(-\frac{a}{\nu}\right)\right] \left[1 - \exp\left(-\frac{b}{\nu}\right)\right]}.$$
(96)

To compute the steady state solution numerically, we integrate the first-order system in time:

$$+ au_x + bu_y = \nu (p_x + q_y),$$

$$p_t = (u_x - p)/T_r,$$

$$q_t = (u_y - q)/T_r,$$

$$(97)$$

until we reach the steady state. On the boundary, in addition to the solution value, we specify also the tangential gradients along on the boundary (simply because they can be computed from the solution on the

$16~{\rm of}~24$



Figure 9. The number of iterations to reduce the nodal residuals by five orders of magnitude. Circles: the upwind advection-diffusion scheme for all values of Re in Table 1. Squares and stars: the scalar Galerkin scheme for Re = 100 and 1000, respectively.

boundary). The advection velocity is fixed as (a, b) = (1.0, 0.8) for this study. The viscosity coefficient, ν , is determined for a given global Reynolds number, $Re = \sqrt{a^2 + b^2}/\nu$. For the length scale, L_r , we use the optimal formula (8) for all cases. The time step is taken as global with CFL = 0.99 for all cases.

VI.B.1. Structured Grids

We consider solving the problem on a series of structured grids of non-uniform spacing (see Figure 3) with N_x points distributed along each line in x-direction and N_y points in y-direction: 17x17, 33x33, and 65x65 grids. Nodes have been generated along each coordinate direction by the same stretching function as in one dimension, i.e., by (93) with $\alpha = 2.0$. We apply the two-dimensional scheme described in Subsection V.A to integrate the advection-diffusion system towards the steady state, for a range of Reynolds numbers: $Re = 10^k$, k = -2, -1.5, -1, -0.5, 0, 0.5, 1, 1.5, 2. The trapezoidal rule was used in the source term discretization to achieve second-order accuracy unless otherwise stated. The method is taken to be converged when the residual is reduced by ten orders of magnitude for all case, in order to ensure that the solutions are fully converged.

The number of iterations required for convergence are given in Table 2. Clearly, the number of iterations is nearly independent of the Reynolds number. Also, observe that we have $O(N^{\frac{1}{2}})$ convergence, even in the diffusion limit, as predicted in Subsection IV.B. A slightly larger iteration number for $Re = 10^2$ on the 17x17 grid is considered due to a tiny oscillation caused by the trapezoidal rule in the source term discretization. In Figure 11, actual L_{∞} errors taken all over the nodes are shown for all three variables, u, p, and q. The results confirm that second-order accuracy has been achieved for all variables for all Reynolds numbers, *including* the gradient variables on the boundary. Again, as in one dimension, the scheme is second-order accurate on the boundary also. We note in Figure 11 that the actual errors go down as the global Reynolds number decreases. This is because the exact solution (96) behaves like xy, i.e., linear in each direction, as diffusion dominates, and the scheme is designed to preserve such linear solutions (residual property).



Figure 10. L_{∞} Errors obtained by the upwind advection-diffusion scheme for the one-dimensional problem for various Reynolds numbers. Second-order accuracy is confirmed for all Reynolds numbers.

To demonstrate the effect of source term discretizations as discussed in Subsection IV.A, we computed the steady solution for Re = 100 on a uniform 17x17 grid with two different source term quadratures: trapezoidal and one-sided. In this setting, we violate the monotonicity condition, $Re \leq 2$ with large cell Reynolds numbers: $ah/\nu \approx 4.88$ and $bh/\nu \approx 3.90$. Results are shown in Figure 12. As expected, we obtained a highly oscillatory solution with the trapezoidal rule as in Figure 12(a), and a monotone solution with the one-sided quadrature as in Figure 12(b).

VI.B.2. Unstructured Triangular Grids

We now demonstrate the second-order accuracy of the scheme developed for triangular grids in Subsection V.B, by applying it to a series of irregular triangular grids: each grid generated from a structured grid with random diagonal splittings and nodal perturbations (see Figure 13). There are seven irregular grids, generated independently, corresponding to the number of nodes: 17x17, 24x24, 33x33, 41x41, 49x49, 57x57, and 66x66. As in the structured grid case, we march in time until the residual is reduced by ten orders of magnitude, with the second-order accurate source term quadrature. Results were obtained for a range of Reynolds numbers: $Re = 10^k$, k = -2, -1.5, -1, -0.5, 0, 0.5, 1, 1.5, 2.

 L_1 error convergence results are shown in Figure 14 for both the solution and the gradients. As can be seen clearly, the scheme is second-order accurate for all variables and for all Reynolds numbers. Note that the solution, u, shows a third-order behavior in the diffusion dominated cases, as mentioned in Subsection V.B. L_{∞} errors show some irregularity but generally go down at second-order accuracy. Figure 15 shows the number of iterations to reach the steady state versus the square root of the number of nodes. It is observed that the number of iterations is proportional to square root of the grid size. Also, the number of iterations

log _{eo} Re	Number of Iterations			
10810 100	17x17	33x33	65x65	
-2.0	812	1864	3980	
-1.5	814	1868	3987	
-1.0	818	1879	4010	
-0.5	833	1911	4081	
0.0	877	1989	4162	
0.5	938	2102	4506	
1.0	894	1948	4034	
1.5	930	1813	3666	
2.0	1358	1966	3837	

Table 2. The number of iterations for the upwind advection-diffusion scheme on structured grids to reduce the nodal residuals by ten orders of magnitude. $Re = \sqrt{a^2 + b^2}/\nu$.

are almost independent of the Reynolds number. Finally, the upwind quadrature in Subsection V.B was tested for a case, Re = 100 on a regular uniform triangular grid with 17x17 nodes. As can be seen in Figure 16, the scheme produced a monotone solution with the upwind quadrature.

VII. Navier-Stokes System

The form of the first-order advection-diffusion system suggests that we solve the Navier-Stokes equations in the form:

$$\partial_t \rho + \partial_x (\rho u) = 0, \tag{98}$$

$$\partial_t(\rho u) + \partial_x(\rho u^2 + p - \mu\tau) = 0, \qquad (99)$$

$$\partial_t(\rho E) + \partial_x(\rho u H - \mu u \tau + \mu q) = 0, \qquad (100)$$

$$\partial_t \tau - \frac{1}{T_v} \left(\frac{4}{3} \partial_x u - \tau \right) = 0, \tag{101}$$

$$\partial_t q - \frac{1}{T_q} \left(-\frac{\partial_x (a^2)}{P_r(\gamma - 1)} - q \right) = 0, \qquad (102)$$

where μ is the viscosity coefficient, P_r is the Prandtl number, $\gamma = 1.4$, $a^2 = \gamma p/\rho$,

$$E = \frac{1}{\gamma - 1} \frac{p}{\rho} + \frac{u^2}{2}, \quad H = \frac{\gamma}{\gamma - 1} \frac{p}{\rho} + \frac{u^2}{2}, \tag{103}$$

 T_v and T_q are the time scales associated with the viscous stress and the heat flux, i.e., τ and q which are now included as additional variables. In two dimensions, there will be 5 additional variables: 3 independent viscous stress components and 2 heat flux components. In three dimensions, there will be 9 additional variables: 6 independent viscous stress components and 3 heat flux components.

Potential benefits of solving the Navier-Stokes equations as a first-order system as above (by whatever discretization methods) include:

• Uniformly accurate schemes made easy: discretizing the whole first-order Navier-Stokes system by one method will ensure a design accuracy for all Reynolds numbers. This offers a very simple solution to discretization methods which suffer from the compatibility issue: a design order of accuracy lost when an inviscid scheme and a viscous scheme are naively added [16–18]. We no longer need to develop two different schemes (inviscid and viscous) and carefully combine them; we just need one scheme for the whole system.



Figure 11. L_{∞} errors obtained for structured grid cases.

- Stencil made compact: the first-order Navier-Stokes system contains only first-derivatives, which can be discretized with nearest-neighbors only. In particular, any sorts of gradient reconstructions (e.g., by least-squares fits) are not required for viscous discretizations, which would largely extend the viscous stencil. This is an obvious advantage for parallelization and Jacobian construction in implicit schemes, etc.
- Various compact viscous discretizations made available: because the viscous part has become hyperbolic, various discretizations are now available: Lax-Wendroff, Lax-Friedrichs, Upwind, etc. As shown in the model problems[1,2], a common Galerkin discretization is just one special case of a family of Lax-Wendroff-type schemes for the first-order diffusion system.

[Alternatively, we may simply ignore updates to the diffusive fluxes, and thus derive a viscous discretization with an explicit diffusive flux reconstruction (the scheme can then be time-accurate with $O(h^2)$ time step). The first-order system approach serves, thereby, also as a tool for deriving (or improving existing) viscous discretizations directly solving the second-derivative viscous terms[1]. Improvements come, for example, in removing spurious error modes (establishing the h-ellipticity[19,20)] and/or enhancing a stability property, through a dissipation term introduced by an advection scheme applied for the first-order system[1]. Also, a great advantage of diffusion schemes derived this way is that they can be implemented in the framework of advection schemes. All mechanisms for advection schemes, such as solution representation, reconstructions, and linearization, are directly applicable to diffusion. A preliminary study shows that a variety of diffusion schemes, including well-known ones, e.g., reconstruction-based finite-volume diffusion schemes or continuous/discontinuous Galerkin diffusion schemes, can actually be derived based on this approach. Details on this alternative approach will be reported elsewhere.]

• Accurate viscous/heat fluxes: diffusive fluxes can be computed simultaneously with the equal accuracy as the main variables; no post-processing, which often causes an reduced order of accuracy

20 of 24



(a) Oscillatory solution with the trapezoidal rule.

(b) Monotone solution with the one-sided quadrature.

Figure 12. Numerical solutions obtained with (a)the trapezoidal rule and (b)the one-sided quadrature in the source term, for Re = 100 on a uniform 17x17 structured grid. z-axis corresponds to the solution, u.

for these quantities, is needed. For numerical methods which store unknowns directly on boundaries, the viscous/heat fluxes will be readily available with a comparable accuracy where they are most needed: a solid body for estimating viscous drag and heating rate.

- O(h) time step: this is a tremendous advantage for explicit time-stepping schemes. Time-marching of a Navier-Stokes code is now as fast as that for an Euler code. In other words, explicit(stationary) iterative schemes for a Navier-Stokes discretization (e.g., Jacobi or Gauss-Seidel) can be as fast as Krylov-subspace methods. For high-Reynolds number flows where the minimum mesh spacing can be too small for explicit time-stepping to be practical, an implicit time-stepping may be employed to go beyond the CFL condition, CFL ≤ 1 , and permit larger CFL numbers. In such a case, O(h) time step implies a fast iterative convergence of a linear system which arises from a linearization of the flow residual.
- **Boundary conditions made simple:** Neumann or Robin boundary conditions turn into the Dirichlet condition through the diffusive flux variables, thus eliminating possibilities of losing accuracy in evaluating derivatives on boundaries.
- Various techniques for hyperbolic systems now applicable: the whole system is now hyperbolic, various techniques are now directly applicable to the Navier-Stokes system through the viscous limit. This includes high-order non-oscillatory schemes, various upwind fluxes, limiters, and much more.

In order to develop upwind schemes (the types of schemes presented here for the model equation) for the first-order Navier-Stokes system, we first determine the time scales, T_v and T_q , and then work out the eigen-structure of the Jacobian matrix, **A**, analyzing the quasi-linear form:

$$\mathbf{U}_t + \mathbf{A}\mathbf{U}_x = \mathbf{Q},\tag{104}$$

where $\mathbf{U} = [\rho, \rho u, \rho E, \tau, q]^t$. There is also a simpler approach where we consider the inviscid and viscous terms separately. For example, for a finite-volume scheme, we may define the dissipation term in the interface flux separately, by solving the inviscid and viscous Riemann problems. Also, for a residual-distribution scheme, we may define a distribution matrix, which distributes a cell-residual of the whole first-order Navier-Stokes system, as a linear combination of an inviscid distribution matrix and a viscous distribution matrix (both of which are upwind), as suggested by (33) and (34). Yet another approach is to employ methods which apply directly to the system without much reference to eigen-structures (e.g., Lax-Friedrich-type methods). Progress made on these developments will be reported in future.



Figure 13. Irregular triangular grid, 17x17 nodes.

Acknowledgments

The author would like to thank Dr. Pieter Buning (NASA Langley) for an illuminating discussion.

References

¹Nishikawa, H., "A First-Order System Approach for Diffusion Equation, I: Second-Order Residual-Distribution Schemes," Journal of Computational Physics, Vol. 227, 2007, pp. 315–352.

 2 Nishikawa, H., "A First-Order System Approach for Diffusion Equation, II: Unification of Advection and Diffusion," to be submitted for publication.

³Roe, P. L. and Arora, M., "Characteristic-Based Schemes for Dispersive Waves I. The Method of Characteristics for Smooth Solutions," *Numerical Methods for Partial Differential Equations*, Vol. 9, 1993, pp. 459–505.

⁴Cattaneo, C., "A Form of Heat-Conduction Equations which Eliminates the Paradox of Instantaneous Propagation," Ct. R. Acad. Sci., Paris, Vol. 247, 1958, pp. 431–433.

⁵Nagy, G. B., Ortiz, O. E., and Reula, O. A., "The Behavior of Hyperbolic Heat Equations' Solutions Near Their Parabolic Limits," *J. Math. Phys.*, Vol. 35, 1994, pp. 4334–4356.

⁶Lowrie, R. B. and Morel, J. E., "Methods for Hyperbolic Systems with Stiff Relaxation," International Journal for Numerical Methods in Fluids, Vol. 40, 2002, pp. 413–423.

⁷Liotta, S. F., Romano, V., and Russo, G., "Central Schemes for Balance Laws of Relaxation Type," *SIAM Journal on Numerical Analysis*, Vol. 38, 2000, pp. 1337–1356.

⁸Jin, S. and Levermore, C. D., "Numerical Schemes for Hyperbolic Conservation Laws with Stiff Relaxation Terms," *Journal of Computational Physics*, Vol. 126, 1996, pp. 449–467.

⁹Arora, M., Explicit Characteristic-Based High-Resolution Algorithms for Hyperbolic Conservation Laws, Ph.D. thesis, University of Michigan, Ann Arbor, Michigan, 1996.

¹⁰Ricchiuto, M. and Nishikawa, H., "A Preliminary Look at Positive Coefficient Residual Distribution Discretizations for Advection-Diffusion," INRIA Report, 2007.

¹¹van der Weide, E. and Deconinck, H., "Positive Matrix Distribution Schemes for Hyperbolic Systems, with Application to the Euler Equations," *Computational Fluid Dynamics 1996*, Wiley, New York, 1996, pp. 747–753.

 12 van der Weide, E., Deconinck, H., Issman, E., and Degrez, G., "A Parallel, Implicit, Multi-Dimensional Upwind, Residual Distribution Method for the Navier-Stokes Equations on Unstructured Grids," *Computational Mechanics*, Vol. 23, 1999, pp. 199–208.

¹³Abgrall, R., "Toward the Ultimate Conservative Scheme: following the quest," *Journal of Computational Physics*, Vol. 167, 2001, pp. 277–315.

¹⁴Nishikawa, H., "Higher-Order Discretization of Diffusion Terms in Residual-Distribution Methods," 34th VKI CFD Lecture Series Very-High Order Discretization Methods, VKI Lecture Series, 2005.

¹⁵Nishikawa, H., Rad, M., and Roe, P., "A Third-Order Fluctuation-Splitting Scheme That Preserves Potential Flow," 15th AIAA Computational Fluid Dynamics Conference, AIAA Paper 01-2595, Anaheim, 2001.

22 of 24



Figure 14. L_1 errors for unstructured triangular grid cases.

¹⁶Nishikawa, H. and Roe, P. L., "On High-Order Fluctuation-Splitting Schemes for Navier-Stokes Equations," *Computational Fluid Dynamics 2004*, Springer-Verlag, 2004, pp. 799–804.

¹⁷Brooks, A. N. and Hughes, T. J. R., "Stabilized finite element methods: I. Application to the advective-diffusive model," *Comp. Meth. Appl. Mech. Engrg.*, Vol. 95, 1992, pp. 253–276.

¹⁸Brooks, A. N. and Hughes, T. J. R., "Streamline Upwind/Petrov-Galerkin Formulations for Convective Dominated Flows with Particular Emphasis on the Incompressible Navier-Stokes Equations," *Comp. Meth. Appl. Mech. Engrg.*, Vol. 32, 1982, pp. 199–259.

¹⁹Brandt, A., Multigrid Techniques: 1984 Guide with Applications to Fluid Dynamics, GMD-Studien, 1984.

²⁰Trottenberg, U., Oosterlee, C. W., and Schüller, A., Multigrid Basics, Parallelism and Adaptivity, Academic Press, 2001.



Figure 15. Number of iterations required for the LDA advection-diffusion scheme to reduce the nodal residuals by ten orders of magnitude for irregular triangular grids.



(a) Oscillatory solution with the trapezoidal rule.



(b) Monotone solution with the upwind quadrature.

Figure 16. Numerical solutions obtained with (a)the trapezoidal rule and (b)the upwind quadrature in the source term, for Re = 100 on a uniform 17x17 triangular grid. z-axis corresponds to the solution, u.