# Uses of Zero and Negative Volume Elements for Node-Centered Edge-Based Discretization 

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#### Abstract

This paper discusses the uses of zero and negative volume elements for the node-centered edge-based discretization. It is shown that the edge-based discretization does not suffer from degraded accuracy nor instability with zero- or even negative-volume elements, and that zero and negative-volume elements can be useful for applications such as discontinuity capturing, singularity resolutions, hanging nodes, and overset grids. Numerical results demonstrate that third-order accuracy can be achieved with zero- and negative-volume elements.


## I. Introduction

Quality of computational grids has been known to greatly impact accuracy and robustness in Computational Fluid Dynamics (CFD) simulations. Over the past decades, there have been a great deal of continuing efforts in improving the quality of unstructured grids, which are typically not of as high quality as structured grids, but better suited for practical simulations involving complex geometries. High-quality unstructured-grid generation is still considered as a difficult task, and must to be performed carefully to avoid zero or negative volume (inverted or negatively oriented) elements since many CFD codes consider such elements as 'invalid', and a simulation will not be performed. Grid generation is, therefore, a major critical step in CFD analyses, and places a great burden on practitioners. In order to enable a rapid turnaround in CFD simulations for complex simulations, unstructured grid generation methods need to be further improved, or numerical schemes/solvers that can produce high-accurate solutions on 'bad-quality' or even 'invalid' grids need to be developed. This paper presents a preliminary attempt in the latter. A new scheme, however, is not developed. We demonstrate that an existing discretization method works on grids with zero and negative volume elements, and more importantly, the use of such elements can be useful.


Figure 1: Taken from Ref.[1]. (a) Exact stream function contours. (b) Converged grid. (c) Numerical solution contours on the converged grid. The Cauchy-Riemann equations for the velocity potential and the stream function are solved for both nodal coordinates and the solution variables by a second-order accurate residualminimization adaptive-grid scheme [1].

A fact that numerical methods have the ability to produce solutions with zero- and negative-volume elements has already been known by researchers. In 1997, the author engaged in the development of an adaptive-grid

[^0]method that solves a target equation for a grid and a solution simultaneously [2], and encountered a case where the method converged on a tangled grid [1]. See Figure 1. The code used did not check negative volumes, and the method converged with no sign of difficulties. It was surprising to find that the grid had been tangled, but the numerical solution looked reasonably good for such a grid. Later, the method was shown to produce perfectly captured discontinuities with zero-volume elements [3]. Although strange and seemingly unreal, these results are quite understandable if we take a geometrical interpretation of solutions of partial differential equations (PDEs). For example, in two space dimensions, a solution of a set of PDEs forms a surface embedded in a higher-dimensional space having both dependent and independent variables as coordinates [4]. A residual defined as a discrete version of the integral of the PDEs can be shown to be a measure of a local error in approximating the solution surface by a triangular element $[2,3,5]$. Therefore, the discrete approximation is valid and can be accurate if a triangular element locally approximates the true solution surface (i.e., tangent to the true surface); it does not matter whether the element is inverted or not. Ref.[3] also shows an implication that the grid quality cannot be judged without reference to a numerical scheme. As shown in Figure 2, a numerical solution exhibits a checkerboard error mode on a 'good' orthogonal grid, but the mode is completely eliminated on a 'bad' grid with a checkerboard nodal perturbation.


Figure 2: Taken from Ref.[3]. (a) A 'good' orthogonal grid around a Joukowsky airfoil and pressure coefficient $C_{p}$ contours computed by a residual-minimization scheme. (b) $C_{p}$ distribution over a Joukowsky airfoil computed on the grid (a). (c) A 'bad' non-orthogonal grid and $C_{p}$ contours, both of which were computed by the residualminimization adaptive-grid scheme. (d) $C_{p}$ distribution obtained on the grid (c). An implication is that a good grid can be a bad grid for a bad scheme and a bad grid can be a good grid for a bad scheme (or a bad scheme can be a good scheme on a bad grid). After all, there is no such thing as a good grid or a bad grid.

Later, in 2014, the author encountered another case with a 2D Navier-Stokes finite-volume solver for a viscous flow over a bump. An irregular triangular was generated with unexpectedly large nodal perturbations, but an implicit solver converged with no sign of difficulties. After the computation was done, it was found that the grid had been totally 'ruined' as in Figure 3(a), but the solution looked, again, reasonably good for such a grid. Compare the Mach contours shown in Figure 3(b) and those obtained on a reference untangled grid in Figure 3(c). See also Figure 3(d) for a reasonable agreement in the pressure distribution over the bump.

CFD folklore has it that researchers have some experience with computations on grids with zero/negativevolume elements $[6,7]$. However, only a very few technical papers exist in the literature $[8,9]$, to the author's knowledge, perhaps because it is not clear if such grids are useful for practical applications and it is almost always possible to fix them to create 'valid' grids. In fact, the author tried to avoid such grids during the development of the adaptive-grid scheme, and never published a formal paper on numerical solutions on tangled grids. The work reported in Refs. [8, 9] presents an interesting attempt to modify a finite-element method for tangled grids. In contrast, we show that no algorithmic modifications are necessary, and grids with zero and negative volume elements can be introduced to bring advantages over conventional grids.

In this paper, we focus on the node-centered edge-based discretization, and demonstrate that it works with zero and negative volume elements and achieves third-order accuracy. Moreover, we demonstrate that zero and negative volume elements can be useful to extend the potential of the node-centered edge-based method. The use of zero-volume elements is not new, and in fact has already been demonstrated for the edge-based method and the residual-distribution method at a boundary node. In Ref.[10], following the residual-distribution method (see Ref.[11] and references therein), an interior stencil is converted to a boundary stencil by collapsing an edge, and thereby a general boundary quadrature formula is derived that preserves third-order accuracy on arbitrary triangular/tetrahedral grids. The collapsed node is then used as a ghost node to specify boundary conditions.


Figure 3: High-Reynolds number viscous flow over a bump at Mach 0.2 on a tangled $80 \times 40$ triangular grid created by nodal perturbations. The grid has 859 negative-volume elements, and 310 negative dual volumes. Newton's method (CFL $\rightarrow \infty$ ) converged to the level of machine zero residuals in 12 iterations for a first-order inviscid scheme with the Roe flux [18] and a damping-term-only viscous scheme with $\alpha=4 / 3$ [19]. A reference solution in (c) is obtained by the same solver on an untangled $80 \times 40$ triangular grid taken from Ref.[20].

The idea is extended here to the interior domain and negative volume elements.
Several examples will be presented to demonstrate the usefulness of zero and negative volume elements. First, negative-volume elements can allow us to easily achieve third-order accuracy by the edge-based scheme on arbitrary-element grids. The edge-based scheme is known to achieve third-order accuracy on simplex grids $[12,13,14,15,16,17]$. Any non-simplex-element grid can be subdivided into triangles or tetrahedra, but the resulting elements may have negative volumes if quadrilateral or hexahedral elements are not strictly convex. This is not an issue if the edge-based scheme works with negative-volume elements. Second, zero-volume elements can better represent a discontinuous solution such as a shock wave. The upwind flux of Roe has the ability to recognize a shock wave across two states. It allows a cell-centered scheme to capture a shock wave across two adjacent cells, virtually with zero thickness. The same can be made possible for node-centered schemes by introducing zero-volume elements across the discontinuity; otherwise a numerically captured discontinuity will always have a finite thickness. Third, a singular point, where the solution (or the gradient) has multiple values at the same location, e.g., a supersonic flow at a corner, can be handled by placing multiple nodes at the same location and triangulating them with zero-volume elements. Fourth, hanging nodes (nodes in the middle of a side/face of an element), can be handled by the edge-based discretization if zero-volume triangles are inserted to form a non-hanging mixed grid. It also achieves third-order accuracy by further subdividing quadrilateral elements into triangles. Fifth, overset grids can be made a single grid by triangulating, with inverted negativevolume elements, the region between two adjacent overset-grid boundaries. It is expected not only to preserve the accuracy of the node-centered edge-based discretization, but also guarantee discrete conservation over the entire domain: the sum of the flux balance over all nodes depends only on the flux on the domain boundaries.

Below, we will study and demonstrate these features for model equations in steady problems. Further details and applications to the Euler and Navier-Stokes equations will be presented in subsequent papers.

## II. Governing Equations

Consider a conservation law:

$$
\begin{equation*}
\partial_{\tau} \mathbf{u}+\partial_{x} \mathbf{f}+\partial_{y} \mathbf{g}=\mathbf{s} \tag{II.1}
\end{equation*}
$$

where $\tau$ is a pseudo time variable, $\mathbf{f}$ and $\mathbf{g}$ are fluxes and $\mathbf{s}$ is a source term. It represents three types of model equations that we use in this study: (1)the linear advection-diffusion equation with scalar solution and fluxes,

$$
\begin{equation*}
\mathbf{u}=u, \quad \mathbf{f}=a u-\nu \partial_{x} u, \quad \mathbf{g}=b u-\nu \partial_{y} u \tag{II.2}
\end{equation*}
$$

where $(a, b)$ is a constant advection vector and $\nu$ is a constant diffusion coefficient; (2) the hyperbolic advectiondiffusion system,

$$
\mathbf{u}=\left[\begin{array}{l}
u  \tag{II.3}\\
p \\
q
\end{array}\right], \quad \mathbf{f}=\left[\begin{array}{c}
a u-\nu p \\
-u / T_{r} \\
0
\end{array}\right], \quad \mathbf{g}=\left[\begin{array}{c}
b u-\nu q \\
0 \\
-u / T_{r}
\end{array}\right] \mathbf{s}=\left[\begin{array}{c}
s \\
-p / T_{r} \\
-q / T_{r}
\end{array}\right]
$$

where $T_{r}=L_{r}^{2} / \nu$ and $L_{r}=1 / \max \left(2 \pi, \nu / \sqrt{a^{2}+b^{2}}\right.$ ) (see Ref.[21]); (3) Burgers' equation with scalar solution and fluxes

$$
\begin{equation*}
\mathbf{u}=u, \quad \mathbf{f}=\frac{u^{2}}{2}, \quad \mathbf{g}=u \tag{II.4}
\end{equation*}
$$

which will be used to study discontinuous solutions. In this paper, we consider steady problems only. Unsteady problems will be considered in a subsequent paper.

## III. Edge-Based Discretization

## III.A. Discretization

The node-centered edge-based discretization for Equation (II.1) is given by

$$
\begin{equation*}
0=-\sum_{k \in\left\{k_{j}\right\}} \phi_{j k}\left|\mathbf{n}_{j k}\right|+\mathbf{s}_{j} V_{j} \tag{III.1}
\end{equation*}
$$

where the pseudo time derivative has been dropped as it is not needed for steady problems, $V_{j}$ is the measure of the dual control volume around node $j$ in the set $\{J\}$ of nodes, $\left\{k_{j}\right\}$ is a set of edge-connected neighbors of $j$, $\phi_{j k}$ is a numerical flux, and $\mathbf{n}_{j k}$ is the scaled directed area vector of the edge $[j, k]: \mathbf{n}_{j k}=\left(n_{x}, n_{y}\right)=\mathbf{n}_{j k}^{\ell}+\mathbf{n}_{j k}^{r}$ (see Figure 4). The unit directed area vector is denoted by $\hat{\mathbf{n}}_{j k}=\left(\hat{n}_{x}, \hat{n}_{y}\right)=\mathbf{n}_{j k} /\left|\mathbf{n}_{j k}\right|$. The numerical flux is computed at the edge midpoint with two solution values linearly reconstructed from $j$ and $k$ with solution gradients computed at nodes by a least-squares (LSQ) fit.

The node-centered edge-based discretization (II.1) is known to be exact for linearly-varying fluxes, and thus second-order accurate on arbitrary simplex-element (triangular/tetrahedral) grids with a linear LSQ fit. However, it is formally first-order accurate on other types of elements unless certain geometrical regularity conditions are satisfied (see Ref.[22] and Appendix E in Ref.[23]). A recent study [12, 13] revealed that the discretization (II.1) can achieve third-order spatial accuracy on arbitrary simplex-element grids if LSQ gradients are exact for quadratic functions, i.e., a quadratic LSQ fit, and the fluxes are linearly extrapolated. The third-order edge-based discretization has been extended to the two-dimensional Navier-Stokes equations in Refs. $[20,24,25]$, and three-dimensional viscous flow problems in Refs.[15, 16, 17, 26].

## III.B. Conventional scheme

Conventional schemes are considered for scalar equations: $\mathbf{u}=u$ and the fluxes are scalar-value functions of $u$. The linear extrapolation is performed to obtain the left and right states at the edge midpoint:

$$
\begin{equation*}
\mathbf{u}_{L}=\mathbf{u}_{j}+\frac{1}{2} \nabla \mathbf{u}_{j} \cdot \mathbf{e}_{j k}, \quad \mathbf{u}_{R}=\mathbf{u}_{k}-\frac{1}{2} \nabla \mathbf{u}_{k} \cdot \mathbf{e}_{j k} \tag{III.2}
\end{equation*}
$$



Figure 4: Dual control volume for the node-centered finite-volume method with scaled outward normals associated with an edge, $\{j, k\}$.
where $\mathbf{e}_{j k}=\left(x_{k}-x_{j}, y_{k}-y_{j}\right), \nabla \mathbf{u}_{j}=\left(\partial_{x} \mathbf{u}_{j}, \partial_{y} \mathbf{u}_{j}\right)$ and $\nabla \mathbf{u}_{k}=\left(\partial_{x} \mathbf{u}_{k}, \partial_{y} \mathbf{u}_{k}\right)$ are linear LSQ gradients of $\mathbf{u}$ computed at the nodes $j$ and $k$, respectively. The vector $\mathbf{e}_{j k}$ is the edge vector, and the unit vector is denoted by $\hat{\mathbf{e}}_{j k}=\mathbf{e}_{j k} /\left|\mathbf{e}_{j k}\right|$. The numerical flux is computed by the sum of the upwind advective flux and the alpha-damping diffusive flux [27]:

$$
\begin{equation*}
\phi_{j k}=\frac{1}{2}\left[\mathbf{h}_{R}+\mathbf{h}_{L}\right]-\frac{1}{2}\left(\left|\mathbf{a}_{n}\right|+\frac{\nu \alpha}{\ell}\right)\left(\mathbf{u}_{R}-\mathbf{u}_{L}\right), \tag{III.3}
\end{equation*}
$$

where $\mathbf{a}_{n}$ is the characteristic speed of the advective term in the direction of $\hat{\mathbf{n}}_{j k}$,

$$
\ell=\left\{\begin{align*}
\left|\mathbf{n}_{j k}\right| & \text { if } \min \left(\left|\mathbf{e}_{j k}\right|,\left|\hat{\mathbf{e}}_{j k} \cdot \hat{\mathbf{n}}_{j k}\right|\right)<1.0 \mathrm{e}-14,  \tag{III.4}\\
\left|\mathbf{e}_{j k} \cdot \hat{\mathbf{n}}_{j k}\right| & \text { otherwise } .
\end{align*}\right.
$$

$$
\begin{align*}
& \mathbf{h}_{L}=\mathbf{f}\left(\mathbf{u}_{L}\right) \hat{n}_{x}+\mathbf{g}\left(\mathbf{u}_{L}\right) \hat{n}_{y},  \tag{III.5}\\
& \mathbf{h}_{R}=\mathbf{f}\left(\mathbf{u}_{R}\right) \hat{n}_{x}+\mathbf{g}\left(\mathbf{u}_{R}\right) \hat{n}_{y} . \tag{III.6}
\end{align*}
$$

The gradients in the fluxes, which are required for the diffusive term, are evaluated by the nodal gradients: $\nabla \mathbf{u}_{j}$ for $\left(\mathbf{f}\left(\mathbf{u}_{L}\right), \mathbf{g}\left(\mathbf{u}_{L}\right)\right)$ and $\nabla \mathbf{u}_{k}$ for $\left(\mathbf{f}\left(\mathbf{u}_{R}\right), \mathbf{g}\left(\mathbf{u}_{R}\right)\right)$. The length scale $\ell$ defined as above allows the implementation of a weak boundary condition, where the edge vector has a zero length and also avoids a trouble with a zero skewness measure (i.e., $\hat{\mathbf{e}}_{j k} \cdot \hat{\mathbf{n}}_{j k}=0$ ), which can happen in polyhedral or agglomerated grids. Note also that the damping term needs to be ignored if $\mathbf{n}_{j k}$ vanishes. For the damping parameter $\alpha$, we take $\alpha=4 / 3$, which is known to achieve fourth-order accuracy on Cartesian grids [27], and provide robust and accurate unstructuredgrid viscous-flow computations in two and three dimensions [28, 29].

## III.C. Hyperbolic scheme

A second-order hyperbolic scheme is considered for the hyperbolic advection-diffusion system. The linear reconstruction is performed as in Equation (III.2), but we upgrade the LSQ gradients for $u$ to a quadratic fit by incorporating a curvature term evaluated with the gradient variables $(p, q)$ (see Ref.[15]). The resulting hyperbolic scheme is known as Scheme-IQ, and second-order but can achieve third-order accuracy in the solution variable $u$ (without enlarging the stencil) when the advective term dominates. The numerical flux is given by the upwind flux for both the advective and diffusive terms:

$$
\begin{equation*}
\phi_{j k}=\frac{1}{2}\left[\mathbf{h}_{R}+\mathbf{h}_{L}\right]-\frac{1}{2} \mathbf{Q}\left(\mathbf{u}_{R}-\mathbf{u}_{L}\right), \tag{III.7}
\end{equation*}
$$

where $\mathbf{Q}$ is the dissipation matrix given by

$$
\mathbf{Q}=\left[\begin{array}{ccc}
\left|a \hat{n}_{x}+b \hat{n}_{y}\right|+a_{v} & 0 & 0  \tag{III.8}\\
0 & a_{v} \hat{n}_{x}^{2} & a_{v} \hat{n}_{x} \hat{n}_{y} \\
0 & a_{v} \hat{n}_{x} \hat{n}_{y} & a_{v} \hat{n}_{y}^{2}
\end{array}\right]
$$

where $a_{v}$ denotes the pure diffusion speed given by

$$
\begin{equation*}
a_{v}=\sqrt{\frac{\nu}{T_{r}}}=\frac{\nu}{L_{r}} \tag{III.9}
\end{equation*}
$$

The hyperbolic scheme is used here to study third-order accuracy of the node-centered edge-based discretization for advection dominated problems, and also for solving the Laplace equation with $(a, b)=(0,0)$. Hyperbolic schemes are known to achieve the same order of accuracy for the solution and the gradients, but when advection dominates, third-order accuracy is achieved for the solution while the gradients remain second-order accurate [21, 30].

## IV. Jacobian-Free Newton-Krylov Solver

The discretization results in a global system of residual equations:

$$
\begin{equation*}
0=-\boldsymbol{\operatorname { R e s }}\left(\mathbf{U}_{h}\right) \tag{IV.1}
\end{equation*}
$$

where $\mathbf{U}_{h}$ denotes a global solution vector for which the system is to be solved. To solve the system, we first consider an implicit defect-correction method:

$$
\begin{equation*}
\mathbf{U}_{h}^{n+1}=\mathbf{U}_{h}^{n}+\Delta \mathbf{U}_{h} \tag{IV.2}
\end{equation*}
$$

where $n$ is the iteration counter, and the correction $\Delta \mathbf{U}_{h}$ is obtained as the solution to the linearized system:

$$
\begin{equation*}
\frac{\partial \widetilde{\mathbf{R e s}^{\prime}}}{\partial \mathbf{U}_{h}} \Delta \mathbf{U}_{h}=-\boldsymbol{\operatorname { R e s }}\left(\mathbf{U}_{h}^{n}\right) \tag{IV.3}
\end{equation*}
$$

where the modified residual $\widetilde{\text { Res }}$ is a global residual vector based on a lower-order spatial discretization. The lower-order spatial discretization is taken to be the first-order version of the residual except for the alphadamping scheme, which is based on the low-order damping-term-only flux:

$$
\begin{equation*}
\phi_{j k}=\frac{\nu \alpha}{\ell}\left(u_{k}-u_{j}\right) . \tag{IV.4}
\end{equation*}
$$

The resulting Jacobian is compact, depending only on the neighbors. The linear system is relaxed by a multicolor Gauss-Seidel method until the linear residual is reduced by one order of magnitude.

To construct a more efficient and robust solver, we use a Jacobin-Free Newton-Krylov (JFNK) solver based the Generalized Conjugate Residual (GCR) method [31] with the above implicit defect-correction solver employed as a variable preconditioner as described in Refs.[32,33]. The GCR projection is performed to reduce the following linear residual:

$$
\begin{equation*}
\frac{\operatorname{Res}\left(\mathbf{U}_{h}^{n}+\epsilon \Delta \mathbf{U}_{h}\right)-\operatorname{Res}\left(\mathbf{U}_{h}^{n}\right)}{\epsilon}=-\boldsymbol{\operatorname { R e s }}\left(\mathbf{U}_{h}^{n}\right) \tag{IV.5}
\end{equation*}
$$

where $\epsilon$ is a small parameter as defined in Ref.[34], by one order of magnitude or to reach the maximum of 10 projections, and then the solution is updated as in Equation (IV.2). For problems considered in this paper, the JFNK solver successfully reduces the residual by eight orders of magnitude within 10 iterations in many cases and 17 in a few remaining cases.

Note that the implicit solver here is not a time-stepping scheme. The pseudo time term has already been dropped and does not exist in the discrete equations. It simply solves a set of discrete equations iteratively. The residual can be multiplied by an arbitrary constant locally at a node (e.g., -1 ), and the implicit solver will still work as long as the Jacobian is constructed consistently with the scaled residual. In fact, the same is true with the pseudo time term and even for unsteady problems if the residual includes pseudo and physical time derivatives.

## V. Edge-Based Discretization with Zero and Negative Volume Elements

Let us begin with a few assumptions on computational grids. It is assumed that a grid is defined by a list of nodes and a list of elements, and each element consists of a set of nodes ordered counterclockwise. A boundary grid is defined by a set of nodes ordered in a consistently manner with the element-node ordering. It is further assumed that any grid with zero- or negative-volume elements is topologically equivalent to a grid with all positive-volume elements. More specifically, it can be mapped, without altering the element-connectivity information, onto a grid with nodes ordered counterclockwise for all elements. This assumption is valid for all grids generated from a positive-volume-element grid by perturbing nodal coordinates. Below, both triangular and quadrilateral elements are considered, but only interior stencils are considered since a boundary stencil is a subset of interior stencils [10].

## V.A. Grid Metrics

Grid metrics required in the edge-based discretization are the dual control volumes and the directed area vectors. These quantities must be computed for a given grid based on the assumption that all elements have positive volumes even if they turn out to be negative. Also, the directed area vector is defined at each edge, and computed as pointing from a node to the other based on the assumption that both elements sharing the edge have positive volumes. In short, algorithms to compute these metrics must not be changed even if an element volume is found to be negative or if the directed area vector is found to point in the opposite direction.

## V.A.1. Signed Volume for Elements


(a) Triangle.

(b) Quadrilateral.

Figure 5: Triangular and quadrilateral elements with local numbering. Nodes are ordered counterclockwise for a positive volume.

Consider a triangular element as shown in Figure 5(a), where the nodes are ordered counterclockwise. The volume of the triangle is computed by

$$
\begin{equation*}
V_{T}=\frac{1}{2}\left[x_{1}\left(y_{2}-y_{3}\right)+x_{2}\left(y_{3}-y_{1}\right)+x_{3}\left(y_{1}-y_{2}\right)\right] \tag{V.1}
\end{equation*}
$$

which can be easily derived by a vector product. It is defined as positive for nodes ordered counterclockwise, or equivalently for the vector product pointing in the positive $z$-direction in the right-handed Cartesian coordinate system. The formula, therefore, gives a signed volume. In the case that the element is inverted, i.e., the node ordering is reversed, it will give a negative value.

The volume of a quadrilateral element as in Figure $5(\mathrm{~b})$ is computed by splitting the element into two triangles and applying the above formula, e.g., as the sum of triangles $[1,2,4]$ and $[2,3,4]$ :

$$
\begin{align*}
V_{Q} & =\frac{1}{2}\left[x_{1}\left(y_{2}-y_{4}\right)+x_{2}\left(y_{4}-y_{1}\right)+x_{4}\left(y_{1}-y_{2}\right)\right]+\frac{1}{2}\left[x_{2}\left(y_{3}-y_{4}\right)+x_{3}\left(y_{4}-y_{2}\right)+x_{4}\left(y_{2}-y_{3}\right)\right]  \tag{V.2}\\
& =\frac{1}{2}\left[x_{1}\left(y_{2}-y_{4}\right)+x_{2}\left(y_{3}-y_{1}\right)+x_{3}\left(y_{4}-y_{2}\right)+x_{4}\left(y_{1}-y_{3}\right)\right] \tag{V.3}
\end{align*}
$$

Again, this formula gives a signed volume: positive and negative if the nodes are ordered counterclockwise and clockwise, respectively. Also, it can be easily shown that the other splitting gives the same result.


Figure 6: Illustrations of triangular grids with positive, zero, and negative volume elements. (a) Stencil with all positive-volume elements. (b) Zero-volume elements, $[j, 1,2]$ and $[j, 6,1]$ (the node $j$ moved to the node 1 ). (c) Negative-volume elements, $[j, 1,2]$ and $[j, 6,1]$ (the node $j$ moved further to the right and slightly above). Note: all elements are defined in the same way, e.g., the element $[j, 1,2]$ is defined by the set of nodes ordered as $j, 1$, and 2 in all cases.

It is important to note that negative volumes should not be made positive and must be left as negative. Any modification to the volume formulas will lead to inconsistency as we will discuss in the next section. The element volumes are used to compute the dual control volume around each node as discussed below.

## V.A.2. Signed Dual Control Volume

The dual control volume is defined at a node $j$ by the area enclosed by the centroids of the surrounding elements and the midpoints of all the edges incident to the node $j$ (See Figure 4). In the case of a triangular element, the contribution to the dual volume is exactly $1 / 3$ of the triangular volume. For a parallelogram, the dual-volume contribution is $1 / 4$ of the quadrilateral volume. Therefore, for a mixed parallelogram-triangular grid, the dual volume is given by

$$
\begin{equation*}
V_{j}=\sum_{T \in\left\{E_{j}\right\}} \frac{V_{T}}{3}+\sum_{Q \in\left\{E_{j}\right\}} \frac{V_{Q}}{4}, \tag{V.4}
\end{equation*}
$$

where $\left\{E_{j}\right\}$ is a set of elements sharing the node $j$, and $T$ and $Q$ denote triangular and quadrilateral elements in the set. For a general quadrilateral grid, the dual volume is computed as a sum of triangular dual-volumes, which are formed over the node $j$, edge-midpoints and the element centroid, over the surrounding elements. For a concave quadrilateral element, the centroid as defined by the arithmetic average of the vertex coordinates may be found outside the element. No modifications should be made even in such a case, and the dual volumes should be computed simply based on the assumption that the centroid is inside the element.

The dual control volume is also a signed quantity, and can be either positive or negative. Note that it can be positive even if some of the elements have negative volumes, and also that it can be nonzero even if some of the elements have zero volumes. For example, the dual volume around the node $j$ is exactly the same (and positive) for all three grids in Figure 6 because terms involving the node $j$ in the dual-volume formula cancel all together when summed over elements sharing the node $j$. That is, the dual control volume around the node $j$ is independent of the coordinates $\left(x_{j}, y_{j}\right)$. One can easily verify this by substituting Equation (V.1) into Equation (V.4). This is true for arbitrary grids: triangular, quadrilateral, and mixed grids. Consequently, the sum of the dual control volumes over all nodes in the domain will result in the total volume of the domain. This is an important property required, for example, for conservation and source term discretizations.

It is possible to have zero dual control volumes. In such a case, the residual vanishes identically, and therefore the solution cannot be determined by solving the residual equation. But such nodes are not necessary, and can be removed from a grid. Or they can be retained and used as a ghost node to implement (external or internal) boundary conditions as described in Ref.[10]. In the rest of the paper, we do not consider zero dual volumes, and assume that the dual volume is nonzero.


Figure 7: Dual face vectors for positive and negative volume elements. Open circles in (b) indicate the midpoint of the nodes $j$ and 2 and the centroid of the collapsed triangle $[j, 1,2]$. Note that the normal vectors are not drawn to scale.

## V.A.3. Directed Area Vector

The directed area vector $\mathbf{n}_{j k}$ is defined at each edge as the sum of the dual face normal vectors associated with the two elements sharing the edge. Consider a triangular element as in Figure 7(a). The dual face normals are computed as

$$
\begin{align*}
& \mathbf{n}_{j 2}^{r}=\left(\frac{y_{j}+y_{2}}{2}-y_{c}, x_{c}-\frac{x_{j}+x_{2}}{2}\right),  \tag{V.5}\\
& \mathbf{n}_{j 1}^{\ell}=\left(y_{c}-\frac{y_{j}+y_{1}}{2}, \frac{x_{j}+x_{1}}{2}-x_{c}\right), \tag{V.6}
\end{align*}
$$

where $\left(x_{c}, y_{c}\right)$ are the centroid coordinates. Exactly the same formulas are used in other cases: zero volume and negative volume elements. Figure $7(\mathrm{~b})$ shows the case of zero volume, created by collapsing the edge $[j, 1]$. The dual face normals do not vanish: $\left|\mathbf{n}_{j 2}^{r}\right|=\frac{1}{6} \Delta l_{12}$ and $\left|\mathbf{n}_{j 1}^{\ell}\right|=\frac{1}{3} \Delta l_{12}$, and both are normal to the edge [1, 2]. Figure 7(c) shows the case of negative volume, which is created from the triangle in Figure 7(a) by moving the node $j$ to the right side of the node 1. In this case, the dual face normals as computed by the formulas (V.5) and (V.6) are directed inward. All these vectors are correct in the sense that the sum of them around a node vanishes:

$$
\begin{equation*}
\sum_{k \in\left\{k_{j}\right\}} \mathbf{n}_{j k}=0 \tag{V.7}
\end{equation*}
$$

which is required for the discretization to be consistent with the target equation. To prove it, split $\mathbf{n}_{j k}$ into the dual face normals and write it as a sum over the elements:

$$
\begin{equation*}
\sum_{k \in\left\{k_{j}\right\}} \mathbf{n}_{j k}=\sum_{E \in\left\{E_{j}\right\}}\left(\mathbf{n}_{j 1}^{\ell}+\mathbf{n}_{j 2}^{r}\right), \tag{V.8}
\end{equation*}
$$

where 1 and 2 are the local node numbers in the element $E$ as in Figure 7. By the formulas (V.5) and (V.6), we obtain

$$
\begin{align*}
\sum_{k \in\left\{k_{j}\right\}} \mathbf{n}_{j k} & =\sum_{E \in\left\{E_{j}\right\}}\left(\frac{y_{j}+y_{2}}{2}-\frac{y_{j}+y_{1}}{2}, \frac{x_{j}+x_{1}}{2}-\frac{x_{j}+x_{2}}{2}\right) \\
& =\sum_{E \in\left\{E_{j}\right\}}\left(\frac{y_{2}-y_{1}}{2}, \frac{x_{1}-x_{2}}{2}\right) \\
& =0 \tag{V.9}
\end{align*}
$$

which is true for all interior stencils. The same is true for quadrilateral and mixed grids and the directed area vectors must be computed in the same way as described above even if the centroid is located outside the
element. Again, it is emphasized that the above identity is true because the dual face normals are directed inward in negative-volume elements (thus they must not be altered). Furthermore, this property is important to guarantee discrete conservation.

It is tempting to argue that the left and right states should be reversed in the upwind flux if the directed area vector points in the opposite direction as in Figure 7 (c) or the scheme will go unstable. However, such a local fix should never be performed; or it will lead to inconsistency and instability. Further details will be discussed in a subsequent paper.

## V.B. LSQ Gradients

The LSQ gradients are computed by fitting a linear function over a set of neighbors. It has nothing to do with elements, and so with zero or negative volumes. Therefore, the LSQ gradients do not fail on grids with zero or negative volumes. However, a care must be taken for a LSQ fit with inverse distance weighting. In case the edge length is zero, the weight needs to be modified (or the edge needs to be ignored) to avoid zero division. In this study, we use an unweighted LSQ fit for all problems.

## V.C. Accuracy of Edge-Based Discretization

It can be shown that the edge-based discretization is exact for linear and quadratic fluxes on arbitrary triangular grids, including zero and negative volume elements, provided the neighbors form a closed loop. A proof will be given in a subsequent paper.

(a) Making a shock stencil.

(b) Zero-volume shock stencil.

Figure 8: Construction of zero-volume shock-stencil. The edge $[j, 3]$ consists of two zero-volume triangles; the same for the edge $[j, 6]$. Open circles are the centroids of two zero-volume triangles. The vector $\mathbf{n}_{L}+\mathbf{n}_{R}$ represents the directed area vector between the nodes $j$ and $j^{\prime}$.

## V.D. Shock Capturing

In node-centered schemes, a numerically-captured shock will always have a finite width corresponding to the width of elements over which the shock is captured. In contrast, cell-centered schemes have a potential for capturing a shock with zero width; this is possible if a cell-interface is aligned with a shock and a numerical flux has the ability to recognize it exactly (e.g., the Roe flux). Such a zero-width shock capturing can be made possible in the edge-based scheme by the use of zero-volume elements.

Consider a stencil shown in Figure 8(a), which illustrates a process of constructing a stencil with zero-volume elements. The edges $\left[3,3^{\prime}\right],\left[j, j^{\prime}\right],\left[6,6^{\prime}\right]$ are collapsed as indicated by arrows, and it leads to the stencil shown in Figure $8(\mathrm{~b})$. Note that the nodes $3^{\prime}, j^{\prime}$, and $6^{\prime}$ still exist, and thus there are two nodes at the same locations indicated as $3, j$, and 6 , and there are four zero-volume triangular elements in the edges $[j, 3]$ and $[j, 6]$. Suppose there exists a shock passing through $[3, j, 6]$, separating two states $u=c_{L}$ and $u=c_{R}$ : the nodes $\left[j^{\prime}, 3^{\prime}, 4,5,6^{\prime}\right]$ have $u=c_{L}$, and the others have $u=c_{R}$. Then, the residual of the first-order edge-based scheme for Burgers' equation at the node $j$ reduces to

$$
\begin{equation*}
\operatorname{Res}_{j}=\left(\left|\bar{a}_{n}\right| \Delta u-\Delta f_{n}\right)\left|\mathbf{n}_{L}+\mathbf{n}_{R}\right| \tag{V.10}
\end{equation*}
$$

where $\mathbf{n}_{L}+\mathbf{n}_{R}=\left(\hat{n}_{x}, \hat{n}_{y}\right)\left|\mathbf{n}_{L}+\mathbf{n}_{R}\right|$, and

$$
\begin{equation*}
\Delta u=c_{R}-c_{L}, \quad \Delta f_{n}=\left(\frac{c_{R}^{2}}{2} \hat{n}_{x}+c_{R} \hat{n}_{y}\right)-\left(\frac{c_{L}^{2}}{2} \hat{n}_{x}+c_{L} \hat{n}_{y}\right), \quad \bar{a}_{n}=\frac{c_{L}+c_{R}}{2} \hat{n}_{x}+\hat{n}_{y} \tag{V.11}
\end{equation*}
$$

This is precisely the Rankine-Hugoniot relation in the direction normal to the shock, and therefore vanishes for two states connected by a shock. The relation is independent of the number of elements around the node $j$. Hence, perfect shock capturing is possible with arbitrary triangulations provided the vector $\mathbf{n}_{L}+\mathbf{n}_{R}$ coincides with the shock normal direction at $j$. Numerical results and further discussions will be given in Section VI.C.

## V.E. Negative Dual Control Volume

Consider the linear advection equation: $(\mathbf{f}, \mathbf{g})=(a u, b u)$. At a node having a negative dual control volume, $V_{j}<0$, we expand the residual in the Taylor expansion and find

$$
\begin{equation*}
\sum_{k \in\left\{k_{j}\right\}} \phi_{j k}\left|\mathbf{n}_{j k}\right|=-\left[\partial_{x}(a u)+\partial_{y}(b u)\right]\left|V_{j}\right|+T E, \tag{V.12}
\end{equation*}
$$

where $T E$ is a truncation error. Therefore, the edge-based discretization approximates the negative of the advection equation. It has no impact on the steady state, and also on unsteady problems since a physical time derivative term will be multiplied by $V_{j}$ and thus is negative also. Treatment of negative dual volumes, however, may not be as straightforward as it seems when negative-dual-volume nodes have neighbors with positive dual volume. Further details and discussions will be given in a subsequent paper.

## VI. Numerical Results

## VI.A. Accuracy Verification

To verify accuracy of the edge-based discretization on grids with zero and negative volume elements, we consider four types of grids as shown in Figure 9: (I) a reference regular triangular grid; (II) a grid with zero-volume elements, which is created from the regular grid by moving the nodes located on the right of the centerline ( $x=0.5$ ) nodes onto the corresponding centerline nodes; (III) a grid with negative-volume elements, which is created by moving some of the nodes outside their own stencil, thus the nodes are located inside a neighbor element; (IV) a fully irregular triangular grid with negative-volume elements, which was created from the regular grid by random diagonal swapping and nodal perturbation. In all grids, the dual volumes are all positive. Four levels of grids with $n \times n$ nodes, where $n=17,33,65,129$, have been independently generated for each grid type, and used for error convergence study. Grids III/IV contain 36/8, 196/49, 900/222, and 3844/867 negative-volume elements in $17 \times 17-, 33 \times 33$-, $65 \times 65$-, $129 \times 129$-node grid, respectively.

We solve the linear advection-diffusion equation with $(a, b)=(0.5,-0.83)$ and $\nu=\frac{\sqrt{a^{2}+b^{2}}}{1000}$ in a square domain for the exact solution [35]:

$$
\begin{equation*}
u(x, y)=\cos (2 \pi \eta) \exp \left(\frac{-2 \pi^{2} \nu}{1+\sqrt{1+4 \pi^{2} \nu^{2}}} \xi\right) \tag{VI.1}
\end{equation*}
$$

where $\xi=a x+b y, \eta=b x-a y$. The solution is strongly imposed at boundary nodes for both conventional and hyperbolic schemes. For the hyperbolic scheme, the component of the gradient variables normal to the boundary is computed by the numerical scheme while the tangential component is strongly imposed (since it is known from the exact solution along the boundary).

Error convergence results are shown in Figure 10. The errors are measured in $L_{\infty}$ norm over all interior nodes, and plotted against the effective mesh size $h_{e f f}$, which is computed as the $L_{1}$ norm of the square root of the magnitude of the dual control volumes. The results show that the hyperbolic scheme achieves third-order accuracy on all grids, while the conventional scheme achieves second-order accuracy on all grids. It may be surprising to see that the error levels are not greatly affected by the grid type. Solution contours are shown in Figure 11. Based on the error convergence results, any irregularity seen in the contours is considered as an artifact created by plotting. At nodes, all errors are comparable as shown in the error convergence plot.

## VI.B. Irregular Quadrilateral Grids Subdivided into Triangles

The edge-based scheme achieves second- and third-order accuracy only on triangular/tetrahedral grids. One way to achieve second- and third-order accuracy on other types of elements is to subdivide the elements
into triangles/tetrahedra. However, it can result in negative volume elements if elements are not convex, e.g., a non-convex quadrilateral. Here, we demonstrate that such negative-volume elements do not necessarily cause problems, and the third-order edge-based scheme can be directly applied to irregular quadrilateral grids arbitrarily subdivided into triangles.

Irregular quadrilateral grids were generated with $n \times n$ nodes, where $n=9,17,33,65,129,256$. See Figure 12 (a) for the grid with $n=33$. All quadrilaterals have positive volumes, but not necessarily convex. Quadrilaterals are subdivided into triangles in such a way that a quadrilateral with nodes $1,2,3,4$, ordered counterclockwise split into two triangles with nodes $1,2,3$, and $1,3,4$. The resulting triangular grids have many elements with negative volumes: $9,24,113,418,1755$, and 6861 negative-volume elements for $n=9,17$, $33,65,129$, and 257 , respectively. Figure $12(\mathrm{~b})$ shows the case of $n=33$. The dual volumes are, however, all positive in the subdivided triangular grids. It is possible to adaptively select the splitting direction to ensure positive volumes of triangles, but we employ the topologically uniform splitting here and demonstrate that the edge-based scheme is not affected by the existence of negative-volume triangles.

The exact solution is the same as in the previous section. The implicit solver converged on all grids within 10 iterations. Error convergence results are shown in Figure 13. As expected, the edge-based scheme is only firstorder accurate on irregular quadrilateral grids, but achieves third-order accuracy on the subdivided triangular grids. It is interesting to observe that the errors are smaller on the two coarse grids on quadrilateral grids than on triangular grids. As shown in Figure 13(b), the solution gradient is inconsistent on irregular quadrilateral grids, but second-order accurate on subdivided triangular grids. The inconsistent behavior is typical for firstorder schemes, but not expected for the hyperbolic scheme as it is known to achieve the same order of accuracy for the solution and the gradients.

Numerical solution contours on the representative quadrilateral and subdivided-triangular grids $(n=65)$ are shown in Figure 14. The solutions are accurate and look very similar despite a significant difference in the order of convergence. A striking difference is observed in solution gradient contours as shown in Figure 15. The inconsistent solution gradient is very inaccurate on irregular quadrilateral grids, and the accuracy does not improve in grid refinement because it is zero-th order accurate (i.e., inconsistent).

## VI.C. Discontinuous Solution

To demonstrate the shock-capturing capability of the edge-based scheme, we consider an oblique shock of Burgers' equation. Two states $u=2.4$ on the left and $u=-1.0$ on the right at $y=0$ create a shock running at a speed $d x / d y=0.7$. The solution is computed in a square domain, $(x, y) \in[0,1.35] \times[0,1]$, where the shock begins at $(x, y)=(0.35,0.0)$ and reaches $(x, y)=(1.05,1.0)$. Dirichlet boundary condition is applied on the left, bottom, and right boundaries. The top boundary is taken as outflow. Two grids are considered. One is an unstructured triangular grid with all positive-volume elements (see Figure16(a)), and the other is a similar triangular grids with zero-volume-elements created along the shock (see Figure 17(a)). Both grids have $17 \times 17$ nodes. For this problem, the first-order conventional scheme is used with initial solutions given by randomly-perturbed exact solutions.

Numerical solution obtained on the positive-volume triangular grid is shown in Figure 16. The shock is captured within three nodes at most, but its path is somewhat irregular as typical for grids not aligned with a shock. On the other hand, for the other grid, the shock is perfectly captured within zero-volume elements as can be seen in Figure 17.

Further investigations are necessary for curved shocks, where the perfect shock-capturing property requires a certain condition on the spacing of nodes along a shock. For practical applications, the shock-capturing property as discussed in Section V.D can be ensured by the Roe flux for the Euler equations, but a shock-fitting technique needs to be available to fully exploit the property. If the nodes are adjustable, a perfect shock-capturing can be made possible with an error committed in the shock path instead of the solution values. Recent progress in shock-fitting techniques as reported in Refs.[36, 37, 38,39 ] shows very encouraging results and these methods could be combined with the edge-based scheme to create a simplified approach for perfect shock-capturing.

## VI.D. Singularity

We consider a reentrant corner singularity, which is an exact solution to the Laplace equation:

$$
\begin{equation*}
u(x, y)=\sqrt{r} \sin \left(\frac{\pi-\theta}{2}\right) \tag{VI.2}
\end{equation*}
$$

where $r=\sqrt{x^{2}+y^{2}}$ and $\theta$ is the angle that increases counterclockwise from the positive $x$-axis. The solution $u$ may be considered as a steam function. It represents a flow coming from the left, makes a 180-degree turn at the origin, and going back to the left. Although the solution is continuous everywhere, the gradients (which correspond to the velocity components) have multiple values at the origin. Due to the singularity at the origin, second-order accuracy is hardly observed in uniform grid refinement. Ref.[40] derives theoretical estimates for the order of discretization accuracy: $O(h)$ in $L_{1}$ norm and $O(\sqrt{h})$ in $L_{\infty}$ norm for a second-order scheme on Cartesian grids. The problem was considered also in Ref.[41] as a model for investigating accuracy issues associated with a trailing edge of an airfoil.

We tried this problem with three types of grids: Grid S-I, Grid S-II, and Grid S-III. Grid S-I is a triangular grid obtained from a Cartesian grid as shown in Figure 18(a). Although not visible, there exist two grid lines along the $x$-axis in $x=[-1,0)$. Hence, the upper and lower parts constitute internal boundaries, and the boundary residuals are formed independently for a node in the upper domain and a node (at the same location) in the lower domain. Grid S-II is constructed by combining the left half of Grid S-I (with the two grid lines along the negative $x$-axis) and a polar grid. At the origin, however, both Grid S-I and Grid S-II have a single node.

Grid S-III is similar to Grid S-II, but has multiple nodes at the origin with zero-volume elements to better resolve the singularity. As shown in Figure 18(c), Grid S-III looks the same as Grid S-II, but they are slightly different near the origin. To show this, the process of generating Grid S-III is illustrated in Figure 19. Grid S-II is generated from the grid shown in Figure 19(a). By moving the boundary nodes towards the negative $x$-axis and the origin as indicated by the red arrows, we obtain the grid with zero-volume elements as in Figure 19(b). For an error convergence study, five levels of grids have been generated for each type: Grid S-I with 85, 297, 1105,4257 , and 16705 nodes; Grid S-II with $57,193,705,2689$, and 10497 nodes; and Grid S-III with 60, 198, 714,2706 , and 10530 nodes.

The hyperbolic scheme was applied to the problem with the solution values specified at boundary nodes and the gradient variables $(p, q)$ are both computed by solving the residual equations at boundary nodes. The JFNK solver converged with no problem, and has found solutions as shown in Figure 20. Comparing the contours, we observe that smoother contours are obtained on Grid S-III near the singular point. Error convergence plots in Figure 21 show that as predicted and observed in Refs.[40,41], on both Grid S-I and S-II, the discretization error converges at the rate $O(h)$ in $L_{1}$ norm and $O(\sqrt{h})$ (or even worse) in $L_{\infty}$ norm. However, higher-order error convergence has been achieved on Grid S-III: $O\left(h^{2}\right) L_{1}$ norm and $O(h)$ in $L_{\infty}$ norm. The results indicate that the ability to solve on grids with co-located multiple nodes has a potential for overcoming degraded accuracy at a singular point.

## VI.E. Hanging Nodes

Adaptive Cartesian-grids allow efficient and easy grid adaptation for practical problems. These grids often introduce hanging nodes, and a numerical scheme needs to be modified to deal with them. It is relatively easy for cell-centered schemes, but may not be so easy for the node-centered edge-based scheme. Moreover, the edge-based scheme can achieve only first-order accuracy on such grids since these are irregular quadrilateral grids.

A series of hanging-node grids are generated from a regular quadrilateral grid of size $n \times n$, where $n=$ $17,33,65,129$, by splitting randomly-chosen elements. See Figure $22(\mathrm{a})$. In order to be able to apply the edge-based scheme, we first introduced zero-volume triangles as illustrated in Figure 23. The resulting grid is a quadrilateral-triangular mixed grid, and it looks exactly as in Figure 22(a). To achieve second/third-order accuracy, these grids are further subdivided into triangles as shown in Figure 22(b).

The implicit solver rapidly converged on these grids within 10 iterations, and successfully produced the solution shown in Figure 24. The solution contours look reasonably accurate, but the discretization error convergence reveals significant differences. Figure 25 shows the error convergence results. First, the conventional scheme gives first-order accuracy on the mixed grids and second-order accuracy on the triangular grids. Then, as shown in Figure 25(b), the gradients are one-order lower accurate as typical in conventional schemes. Second, the hyperbolic scheme achieves first-order accuracy on the mixed grids with unexpected zero-th order gradients, and third-order accuracy on the triangular grids with second-order gradients as expected.

This example demonstrates that the node-centered edge-based scheme can be directly applied to hangingnode grids with zero-volume triangular elements inserted. Moreover, third-order accuracy can be obtained by further subdividing quadrilateral elements triangles. These grid modifications can be performed inside a code for a given hanging-node grid, and a third-order solution can be returned on the given grid to a user.

## VI.F. Unstructured Overset Grids

To demonstrate that negative-volume elements can be useful, we consider overset (or Chimera) grids. Overset grid approach has been widely used to simulate moving body problems, where a computational grid over a geometrically complex domain is constructed by overlapping grids independently generated around geometrically simple parts of the domain. One of the known issues in the overset method is the violation of discrete conservation. Discrete conservation is trivially satisfied in finite-volume methods, but it does not apply to overset grids because it is not applicable between disjoint grids.

Discrete conservation can be satisfied easily if the gap among overlapping grids is filled with elements to create a single grid and a conservative discretization is applied over the entire domain. This is possible if negative-volume elements are allowed. To illustrate the idea, we constructed an overset grid as shown in Figure 26(a). The upper and lower grids are overlapped, but they are actually connected by triangulation. This is visualized in Figure 26(b) by assigning $z=0$ at nodes on the upper grid, and $z=1$ at nodes on the lower grid. Observe that they are connected by triangular elements to form a single grid. The connecting elements are inverted elements with negative volumes in $(x, y)$ plane. To avoid mixed-positive-negative elements around each node, the top/bottom grid and the negative-element grid are connected via a layer of zero-volume triangular grids. Hence, each node is shared by non-negative-volume element on the top and bottom grids, and by non-positive-volume elements on the negative-dual-volume grid. Now that we have a single grid, the edge-based discretization can be directly applied to the entire grid. No interpolation is necessary between the disjoint grids and the discrete conservation is trivially satisfied since there exists just one grid.

We solve the advection-diffusion problem as in Section VI.A with $\nu=\frac{\sqrt{a^{2}+b^{2}}}{10^{6}}$ by using five levels of connectedoverset grids. The implicit solver converged within 10 iterations, and produced very smooth solutions as shown in Figure 27. Figure 27(a) shows contours on the second grid; it is very smooth despite the complex structure of the overset grid. To further demonstrate the quality of the solution, contours are shown separately for the upper and lower grids in Figure 27(b) and 27(c), respectively. Error convergence results in Figure 28 demonstrate that second-order accuracy is achieved by the conventional scheme and third-order accuracy is achieved by the hyperbolic scheme.

The example shown here should be taken as preliminary. It should be further studied for more realistic boundary conditions. Also, it is not clear if unsteady problems can be solved on such a grid. A different singlegrid construction may need to be explored for more realistic problems. Nevertheless, results are encouraging and potential benefits warrant further investigation.

## VII. Concluding Remarks

We have shown that the node-centered edge-based method directly applies to grids with zero and negative volume elements, and such elements can be introduced to achieve third-order accuracy on general quadrilateral grids by a simple subdivision, to capture a straight shock exactly, to resolve an issue in a singular point, to handle hanging nodes and achieve third-order accuracy, and to achieve discrete conservation and third-order accuracy for overset grids. Numerical results have demonstrated the usefulness of zero and negative volume elements in these examples for model equations. Further studies are necessary to fully understand the uses of these elements, including a theoretical proof of third-order accuracy on grids with zero and negative volume elements, effects of boundary conditions, shock-capturing for curved shocks, unsteady problems, and a deeper investigation into negative and mixed positive-negative dual-volume grids. Finally, the node-centered edge-based scheme is not the only one that works on zero and negative volume elements. For example, the $P_{1}$ continuous Galerkin method for diffusion can be easily made to work with such elements. Investigations for other discretization methods are also left as future work.

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(a) I: Regular triangular grid.

(c) III: Regularly-perturbed negative volume elements.

(b) II: Zero volume elements.

(d) IV: Irregularly-perturbed negative volume elements.

Figure 9: Coarsest grids used for accuracy verification, which all have $17 \times 17$ nodes: (a) Grid I: A reference regular triangular grid, (b) Grid II: Zero-volume element grid created from the regular grid by collapsing the elements on the right side of the centerline $x=0.5$, (c) Grid III: Negative-volume element grid created from the regular grid by moving nodes into the neighbor elements, but the dual volumes are all positive, (d) Grid IV: Negative-volume element grid created by random nodal perturbations.


Figure 10: $L_{\infty}$ discretization error convergence for the solution $u$.


Figure 11: Solution contours on the coarsest grids with $17 \times 17$ nodes:


Figure 12: Irregular quadrilateral and triangular grids. All quadrilaterals have positive volumes, but not necessarily convex. The triangular grid was generated from the irregular quadrilateral grid by subdividing quadrilateral elements, resulting in 113 negative-volume triangles.


Figure 13: Error convergence on irregular quadrilateral and triangular grids.


Figure 14: Solution contours on irregular quadrilateral and triangular grids.


Figure 15: Solution gradient $\left(p=\partial_{x} u\right)$ contours on irregular quadrilateral and triangular grids.


Figure 16: Burgers' equation: Oblique shock on a positive-volume unstructured grid.


Figure 17: Burgers' equation: Oblique shock on a shock-aligned zero-volume-element grid.


Figure 18: Grids used for the reentrant-corner singularity problem. Each grid is the second coarsest grid in a series of six levels of grids.


Figure 19: Making of a zero-volume-element grid (Grid S-III) for the reentrant-corner singularity problem. The final grid has two nodes at the left red point and six nodes at the right red point. Observe that long and thin triangles on the left have become zero-volume elements on the right. Merging multiple nodes at the origin results in Grid S-II.


Figure 20: Solution contours for the reentrant-corner singularity problem obtained on the third finest grid in a series of five levels of grids for each grid type.


Figure 21: $L_{1}$ and $L_{\infty}$ discretization error convergence for the reentrant-corner singularity problem.


Figure 22: Grids used for the reentrant-corner singularity problem. Each grid is the coarsest grid in a series of four levels of grids.


Figure 23: Making of a zero-volume-element hanging-node grid. There are two nodes at the point indicated in red.


Figure 24: Solution contours for hanging-node grids.


Figure 25: Error convergence results for hanging-node grids.


Figure 26: Overset-type grid: (a) 2D view in $(x, y)$ plane; (b) 3D view with $z$ coordinates given 1 in the lower part of the grid, 0 in the upper part, and linearly interpolated between them.


Figure 27: Solution contours on the overset grid: (a) entire overset -grid; (b) top part of the overset -grid; (c) bottom part of the overset-grid. Negative-volume elements are included in (a), not in (b) and (c).


Figure 28: $L_{\infty}$ error convergence results for overset grids.


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