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An Implicit Gradient Method for Cell-Centered Finite-Volume Solver on Unstructured Grids

$$\mathbf{M}_{jj}\mathbf{g}_j + \sum_{k\in\{k_j\}} \mathbf{M}_{jk}\mathbf{g}_k = GG$$

AIAA2019-1155

This work was supported by the Hypersonic Technology Project, through the Hypersonic Airbreathing Propulsion Branch of the NASA Langley Research Center, under Contract No. 80LARC17C0004

Hiroaki Nishikawa, *National Institute of Aerospace*

AIAA SciTech 2019, January 9, 2019

Hiro's CFD AlgorithmShack

Hyperbolic Navier Stokes

NEW! Hyperbolic Method

Hyperbolize it now!

Limiter functions

HNS

Fast and Accurate Hyperbolic Navier-Stokes!

Fast and Accurate Hyperbolic Navier-Stokes!

Hyperbolic!

AlgorithmSha

Hyperbolic Method

Hyperbolize and discretize it. JCP2007 ~

$$\frac{\partial u}{\partial t} = \nu(\partial_{xx}u + \partial_{yy}u)$$

First-order system

$$\frac{\partial u}{\partial t} = \nu(\partial_x p + \partial_y q)$$

$$0 = \partial_x u - p$$

$$0 = \partial_y u - q$$

↓ *Discretize*

$$\frac{du_j}{dt} = Res_j$$

Many existing schemes are here.

First-order hyperbolic system

$$\frac{\partial u}{\partial \tau} = \nu(\partial_x p + \partial_y q) - \frac{\partial u}{\partial t}$$

$$\frac{\partial p}{\partial \tau} = \frac{1}{T_r}(\partial_x u - p)$$

$$\frac{\partial q}{\partial \tau} = \frac{1}{T_r}(\partial_y u - q)$$

↓ *Discretize by upwind*

$$\frac{d}{dt}\begin{bmatrix} u_j \\ 0 \\ 0 \end{bmatrix} = \mathbf{Res}_j'$$

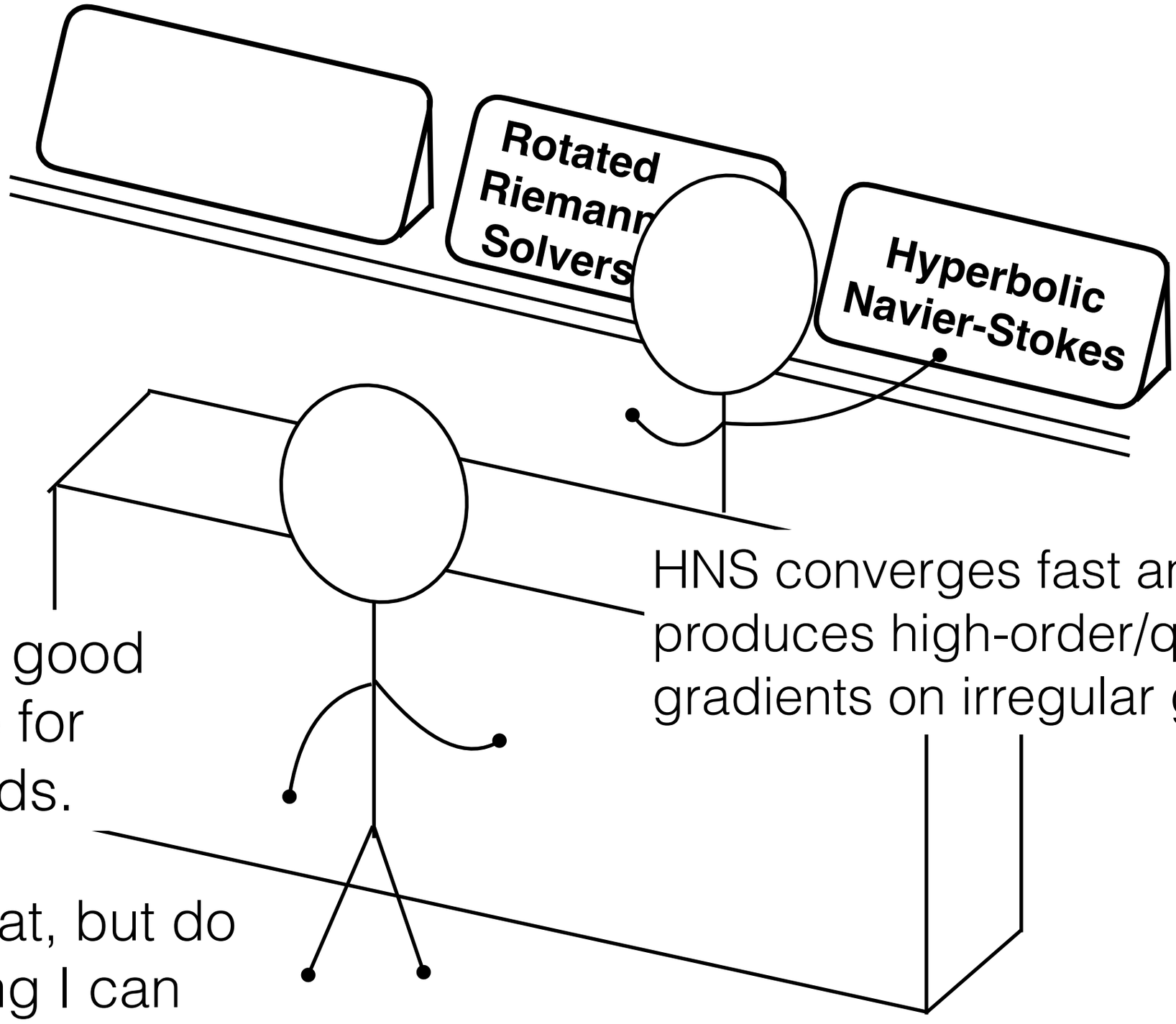
$$\frac{\partial u}{\partial \tau} = \frac{\partial p}{\partial \tau} = \frac{\partial q}{\partial \tau} = 0$$

Superior discretization via hyperbolic system:
(1)high-order gradients, (2)convergence acceleration, (3)high-order advection/inviscid.

Extended to 3D Navier-Stokes — Hyperbolic Navier-Stokes (HNS) AIAA2016 ~

DG/rDG HNS solver: Just presented a minute ago. See AIAA2019-1150

Customers want robust viscous schemes but something simpler to implement...



I'm looking for a good viscous scheme for unstructured grids.

HNS converges fast and produces high-order/quality gradients on irregular grids!

HNS sounds great, but do you have anything I can implement quickly?

Alpha-Damping Scheme AIAA2010-5093

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Extract a conventional (scalar) scheme from hyperbolic scheme.

First-order hyperbolic system

$$\frac{\partial u}{\partial \tau} = \nu(\partial_x p + \partial_y q) - \frac{\partial u}{\partial t}$$

$$\frac{\partial p}{\partial \tau} = \frac{1}{T_r}(\partial_x u - p)$$

$$\frac{\partial q}{\partial \tau} = \frac{1}{T_r}(\partial_y u - q)$$

→ Hyperbolic scheme

$$\frac{du}{dt} = Res'(1)$$

$$0 = Res'(2)$$

$$0 = Res'(3)$$

→ Extracted scheme

$$\frac{du}{dt} = Res'(1)$$

(p, q) = LSQ gradient

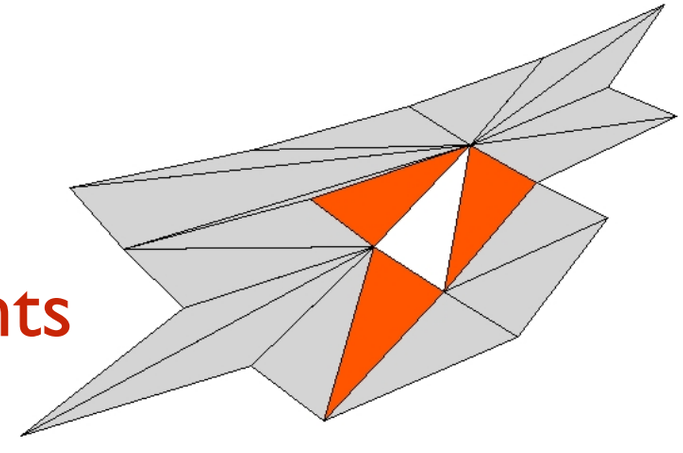
Derivation is applicable to various discretization methods (FV/DG/SV/RD), and gave birth to a new class of adjustable diffusion/viscous schemes for unstructured grids.

See AIAA2010-5093, AIAA2011-3044, JCP2017 for details.

Alpha-damping diffusion/viscous scheme is being used in commercial CFD codes SC/Tetra (CFDS28-2014) and scFLOW, and NASA's VULCAN code (JANNAF-2017), etc.

Many customers want robust gradient methods…

Issues on gradients for finite-volume methods:



- Instability on triangles/tetrahedra if LSQ gradients are computed with face-neighbors

Diskin&Thomas AIAAJ2011 Haider et.al. Numer. Math. 2009

Larger stencil tends to stabilize a solver, but not guaranteed…

- Instability with accurate gradients for inviscid terms.

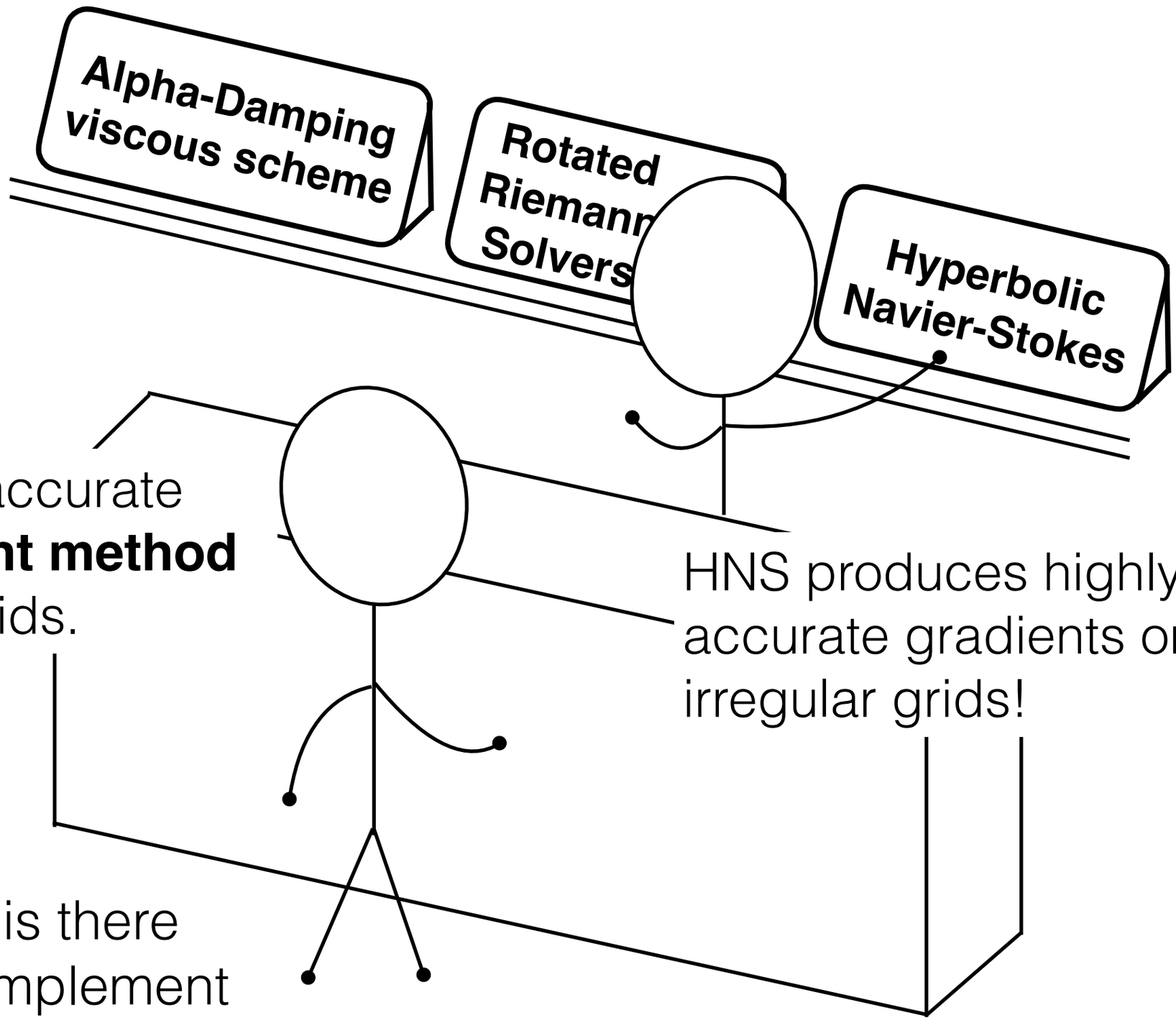
Diskin&Thomas AIAAJ2011

- Instability with inaccurate gradients for turbulence-model source terms.

Diskin&Thomas NIA2008

- Inaccuracy on high-aspect-ratio and curved thin grids. Mavriplis AIAA2003

Another approach was presented in AIAA2019-0127 (efficient stencil augmentations, NASA's VULCAN CFD codes).



I'm looking for an accurate and robust **gradient method** for unstructured grids.

HNS produces highly accurate gradients on irregular grids!

HNS is great, but is there something I can implement quickly?

Implicit Green-Gauss Gradient

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H. Nishikawa, "From Hyperbolic Diffusion Scheme to Gradient Method: Implicit Green-Gauss Gradients for Unstructured Grids", Journal of Computational Physics, Volume 372, 2018, Pages 126-160

Extract a gradient scheme from a hyperbolic diffusion scheme.

First-order hyperbolic system

$$\frac{\partial u}{\partial \tau} = \nu(\partial_x p + \partial_y q) - \frac{\partial u}{\partial t}$$

$$\frac{\partial p}{\partial \tau} = \frac{1}{T_r}(\partial_x u - p)$$

$$\frac{\partial q}{\partial \tau} = \frac{1}{T_r}(\partial_y u - q)$$

Hyperbolic scheme

$$\frac{du}{dt} = Res'(1)$$

$$0 = Res'(2)$$

$$0 = Res'(3)$$

Implicit Gradient scheme

Solution u is given.

$$0 = Res'(2)$$

$$0 = Res'(3)$$

Leading to a global linear system for gradients.

$$\mathbf{M}_{jj}\mathbf{g}_j + \sum_{k \in \{k_j\}} \mathbf{M}_{jk}\mathbf{g}_k = GG$$

$$\mathbf{g}_j = (\partial_x u_j, \partial_y u_j)$$

RHS is the Green-Gauss gradient —> Implicit Green-Gauss (IGG).

Advantages of IGG gradient

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$$\mathbf{M}_{jj}\mathbf{g}_j + \sum_{k\in\{k_j\}} \mathbf{M}_{jk}\mathbf{g}_k = GG$$

(1) Stabilizes a FV solver with many neighbors: Gradient stencil spans the entire grid.

Wang, et. al. JCP2017: Variational Reconstruction. LSQ on RHS.

(2) Exact for linear functions: first-order accurate on irregular grids (higher-order possible).

(3) One relaxation is cheap and compact: only face neighbors, low storage (< LSQ coeffs).

Gauss-Seidel relaxation for IGG:

$$\mathbf{g}_j^{n+1} = \mathbf{g}_j^n + \omega_g \left[\sum_{k\in\{k_j\}} \left\{ -\mathbf{C}_{jk}\mathbf{g}_k^n + \mathbf{c}_{jk}(u_j + u_k) \right\} - \mathbf{g}_j^n \right] \qquad 0 \le \omega_g \le 1$$

[Advantages demonstrated by Wang et. al. JCP2017 with a different implicit gradient method, called variational reconstruction (VR).]

In this work, we investigate IGG for Euler/Navier-Stokes finite-volume solvers.

IGG is Nice and Adjustable

Nishikawa JCP2018

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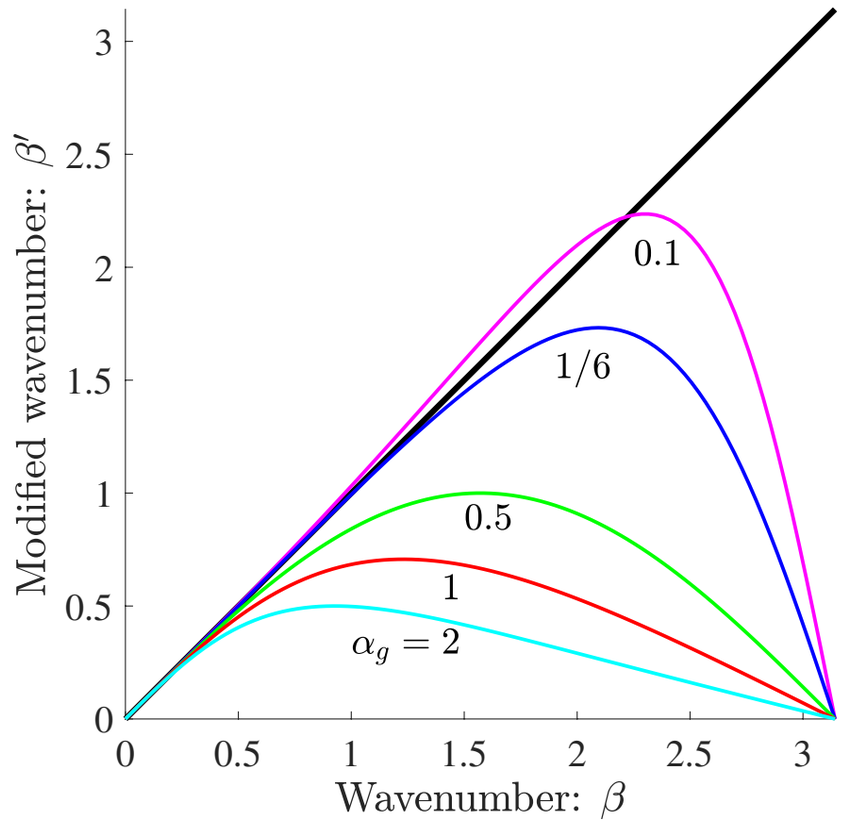
A parameter α_g is inherited from the upwind dissipation.

$$\mathbf{M}_{jj}(\alpha_g)\mathbf{g}_j + \sum_{k\in\{k_j\}} \mathbf{M}_{jk}(\alpha_g)\mathbf{g}_k = GG$$

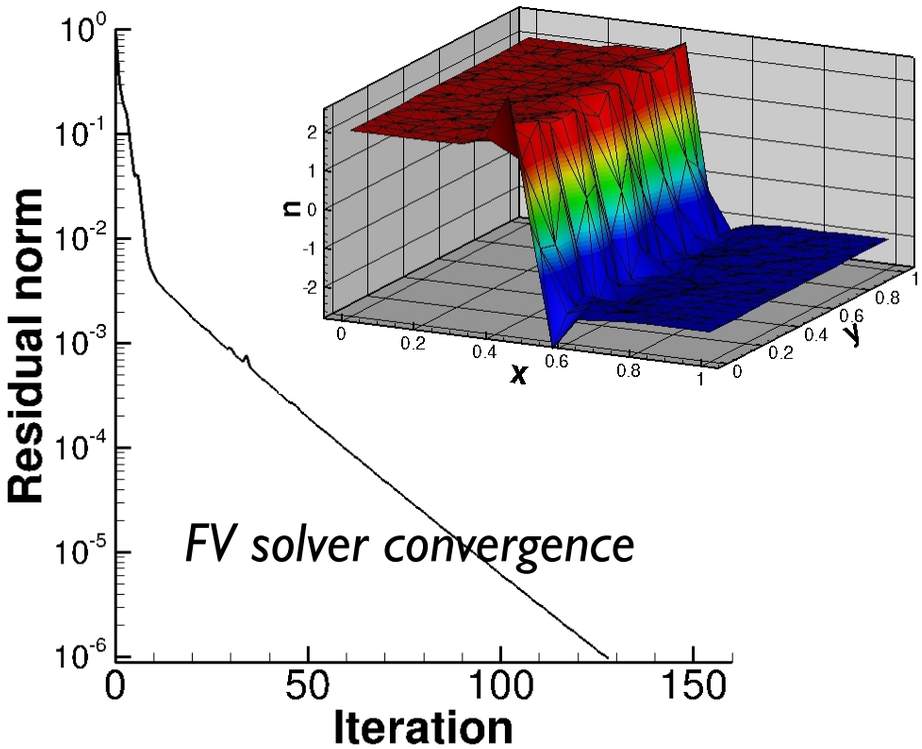
(1) Resolution improves with decreasing α_g, and 4th-order accuracy is achieved at 1/6 through boundaries on regular quadrilateral grids.

IGG (and VR) is an extension of compact schemes to unstructured grids.



(2) Large α_g helps FV solver by damping out the gradient for discontinuous solutions:

Facilitate limiting and convergence (gradient is already small



Good to have an adjustable parameter if you know how to choose its value: 1/6 for 4th-order accuracy on regular grid, a large (e.g., 5) for discontinuities.

See JCP2018 for details.

Hyperbolic Diffusion/Viscous System

Discretize it by upwind schemes.

$$\frac{\partial u}{\partial \tau} = \nu(\partial_x p + \partial_y q) - \frac{\partial u}{\partial t} \quad \frac{\partial p}{\partial \tau} = \frac{1}{T_r}(\partial_x u - p) \quad \frac{\partial q}{\partial \tau} = \frac{1}{T_r}(\partial_y u - q)$$

Hyperbolic Diffusion/Viscous Scheme: $\frac{d\mathbf{U}_j}{d\tau} = \mathbf{Res}_j$

Solve for (p_j, q_j) for a given u_j

Solve for (u_j, p_j, q_j)

Solve for u_j with, e.g., $(p_j, q_j) = LSQ(u)$

Gradient method Implicit Green-Gauss (IGG)

$$0 = \mathbf{Res}_j(2) \approx \partial_x u - p$$
$$0 = \mathbf{Res}_j(3) \approx \partial_y u - q$$

JCP2018, AIAA2019

- Implicit method for gradients
- Extended compact scheme
- Adjustable parameters
- 4th-order through boundary
- Accurate for distorted grids
- Stabilize FV solvers

$$\frac{du_j}{dt} = \mathbf{Res}_j(1)$$
$$0 = \mathbf{Res}_j(2)$$
$$0 = \mathbf{Res}_j(3)$$

JCP2007, 2010, 2012
AIAA papers 2009-2018

- Coupled system scheme
- Highly accurate gradients
- Same accuracy order: u, p, q.
- dtau=O(h), Jac=O(1/h)
- Fast convergence
- Higher-order inviscid scheme
- Implemented in SC/Tetra, FUN3D

Diffusion/Viscous Scheme: *alpha-damping scheme*

$$\frac{du_j}{dt} = \mathbf{Res}_j(1) \approx \nu(\partial_{xx} u + \partial_{yy} u)$$

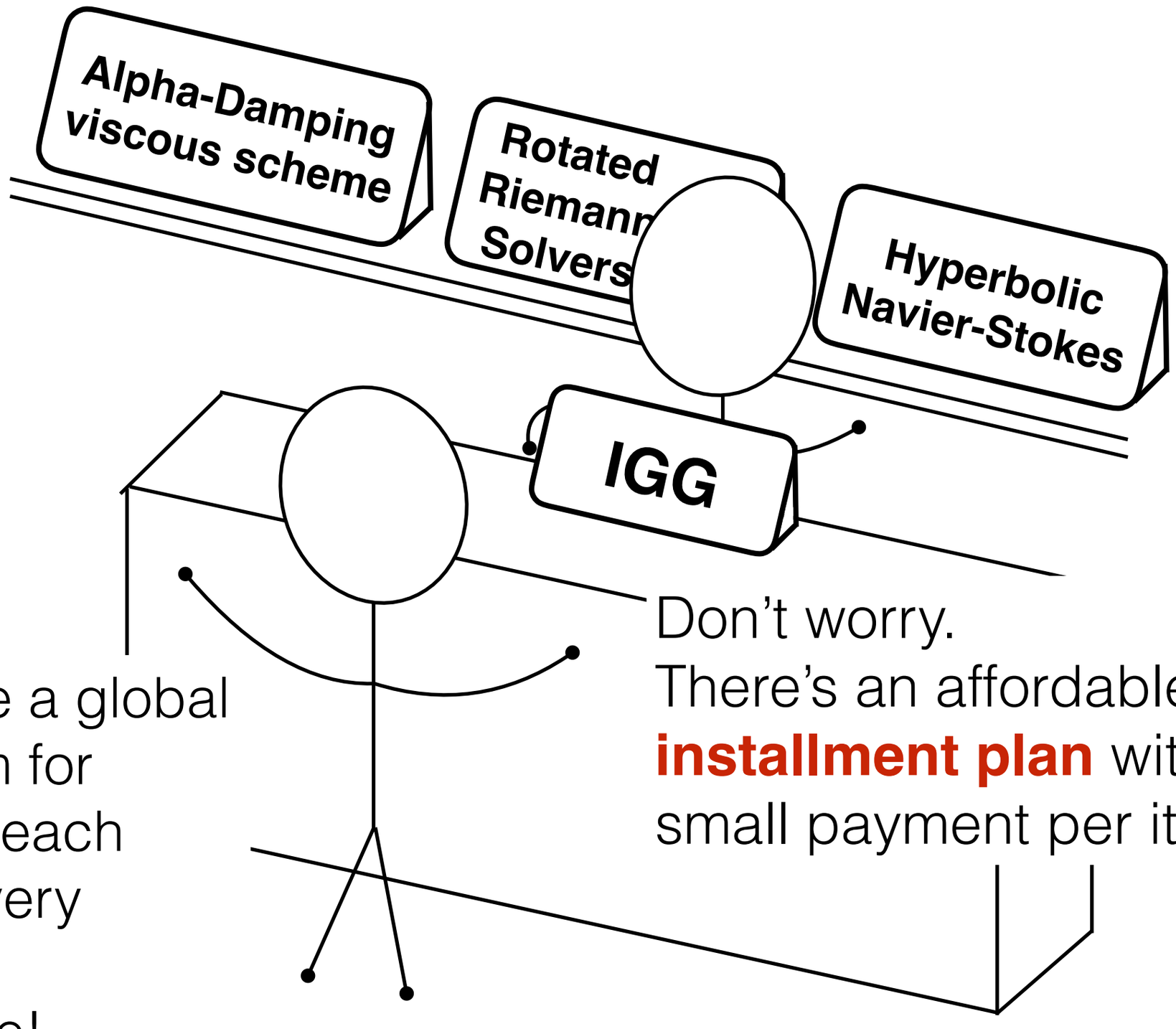
AIAA2010-5093, C&F2011, AIAA2011-3044, JCP2017

- Scalar diffusion scheme
- (p,q) one-order lower than u.
- dt=O(h^2), Jac=O(1/h^2)
- Consistent and damping terms
- Adjustable damping coefficient α
- Used in SC/Tetra, scFLOW, and NASA's VULCAN code.

Affordability matters…

Expensive algorithms hardly sell.

Supercomputer is not available to all CFD users…



Need to solve a global linear system for gradients for each variable at every iteration?
Too expensive!

Don’t worry.
There’s an affordable **installment plan** with a very small payment per iteration!

Three Plans available

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IDC: Implicit defect-correction Navier-Stokes solver with 1st/0th-order residual Jacobian.

(1) IDC solver with IGG

Solve:

$$\mathbf{M}_{jj}\mathbf{g}_j + \sum_{k\in\{k_j\}} \mathbf{M}_{jk}\mathbf{g}_k = GG^n$$

Relax: $\mathbf{J}\Delta\mathbf{U} = -\mathbf{Res}(\mathbf{U}^n, \mathbf{g})$

Update: $\mathbf{U}^{n+1} = \mathbf{U}^n + \Delta\mathbf{U}$

Solve IGG system at every iteration - "Too expensive!"

(2) IDC-IGG solver

Relax:

$$\mathbf{M}_{jj}\mathbf{g}_j^{n+1} = GG^n - \sum_{k\in\{k_j\}} \mathbf{M}_{jk}\mathbf{g}_k^n$$

Relax: $\mathbf{J}\Delta\mathbf{U} = -\mathbf{Res}(\mathbf{U}^n, \mathbf{g}^{n+1})$

Update: $\mathbf{U}^{n+1} = \mathbf{U}^n + \Delta\mathbf{U}$

Iterate for IGG gradients along with the IDC solver.
As cheap as LSQ!

(3) Coupled Newton solver

Coupled residual and Jacobian
- 20x20 block for 3D NS.

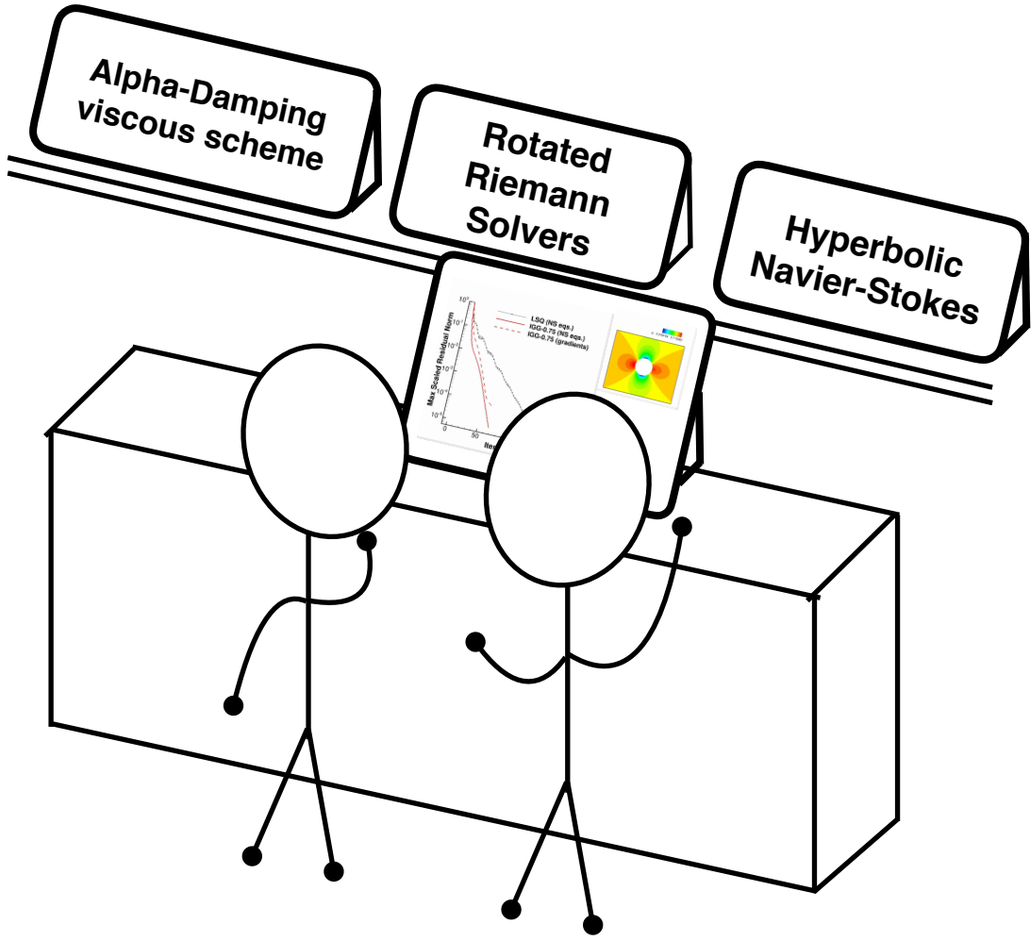
Relax: $\mathbf{J}^{exact}\Delta\mathbf{U} = -\mathbf{Res}(\mathbf{U}^n)$

Update: $\mathbf{U}^{n+1} = \mathbf{U}^n + \Delta\mathbf{U}$

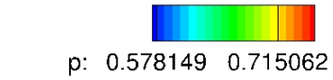
Requires more memory and cost per iteration, but very robust.

In this paper, we consider Plan (2) for Euler and Navier-Stokes computations.

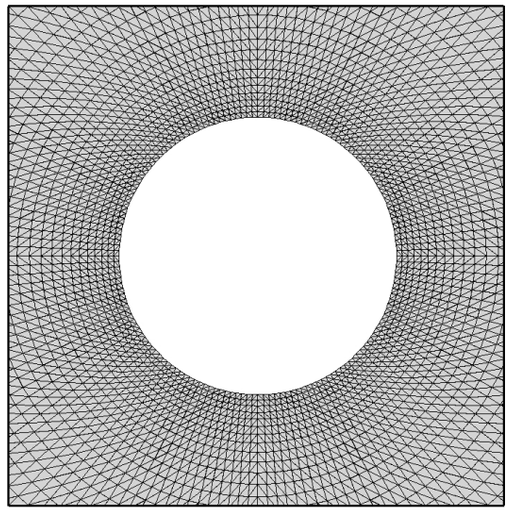
Now start a sales pitch.

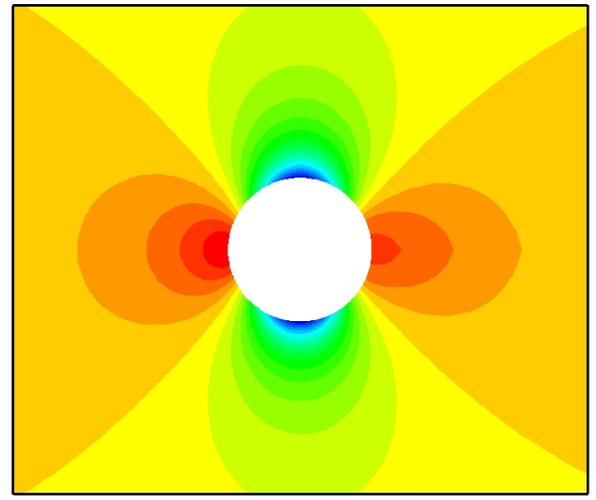


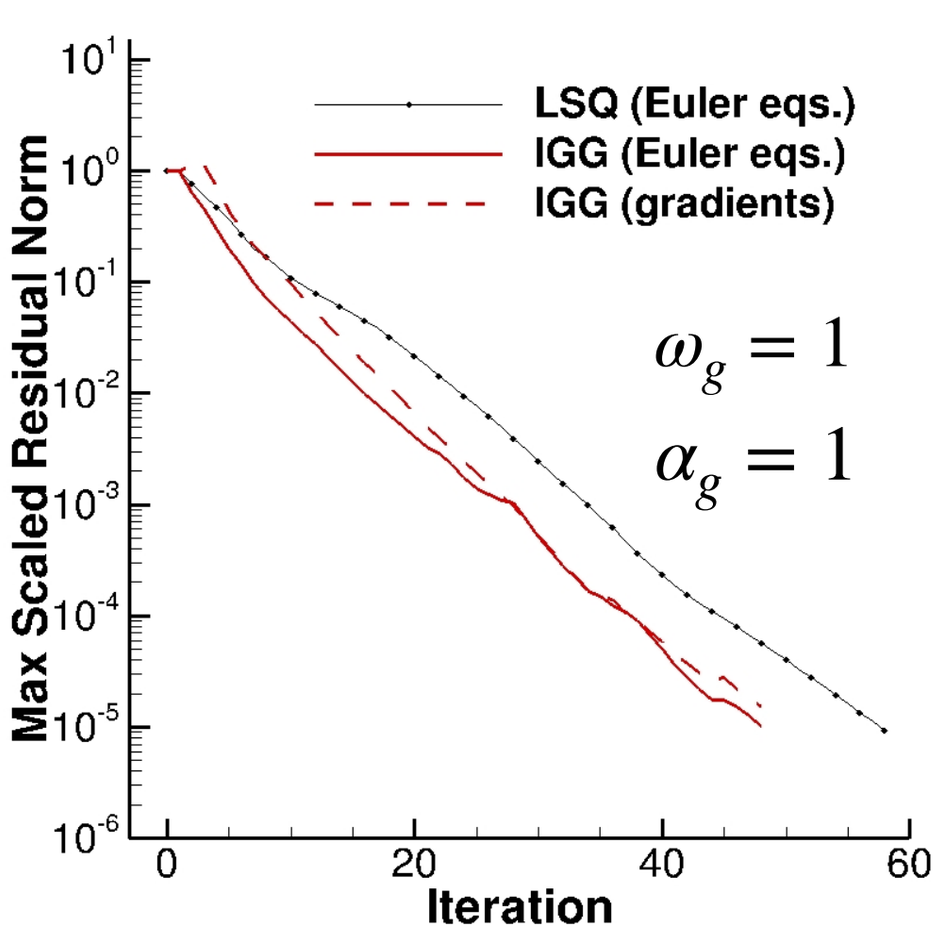
Subsonic Cylinder at Mach=0.3 (Roe flux)

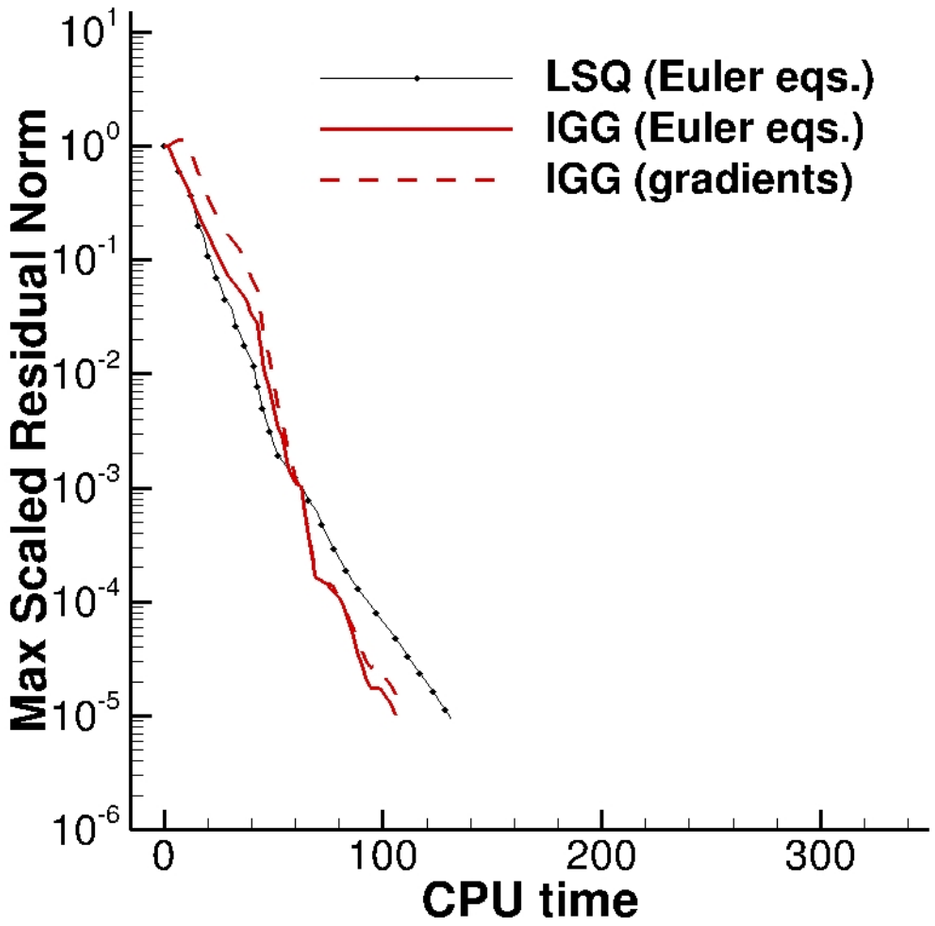


LSQ with vertex neighbors: all neighbors sharing vertices of a target cell.



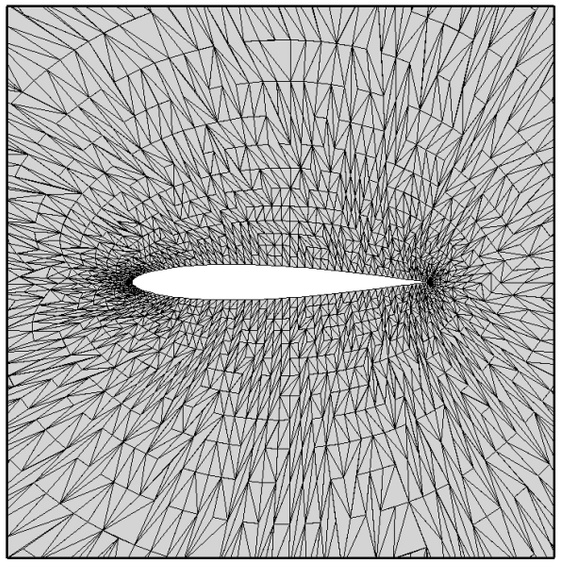


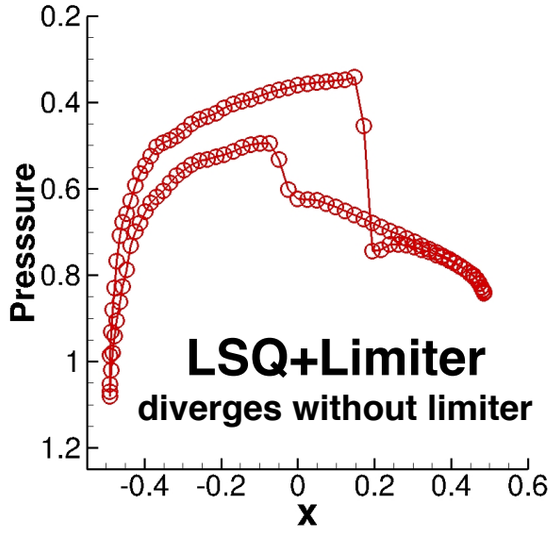


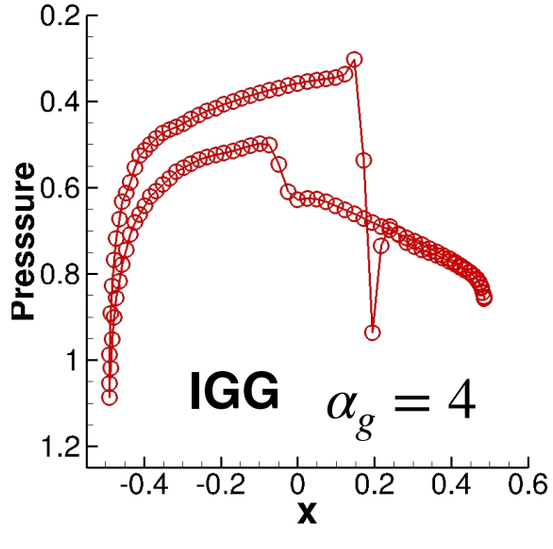


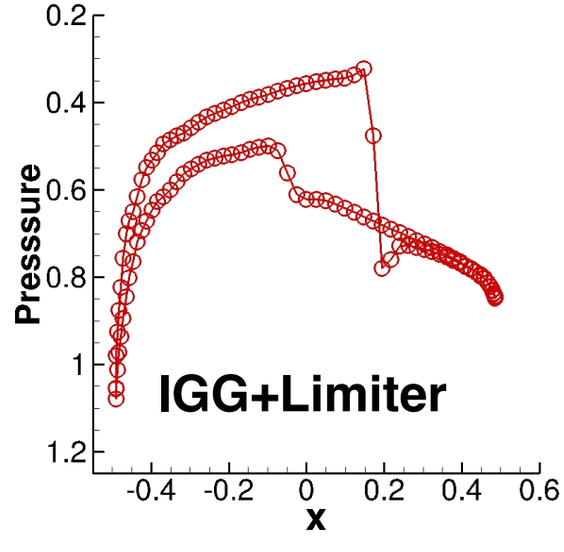
IGG is not more expensive than LSQ gradients

Transonic Airfoil at Mach=0.8 (Roe flux)

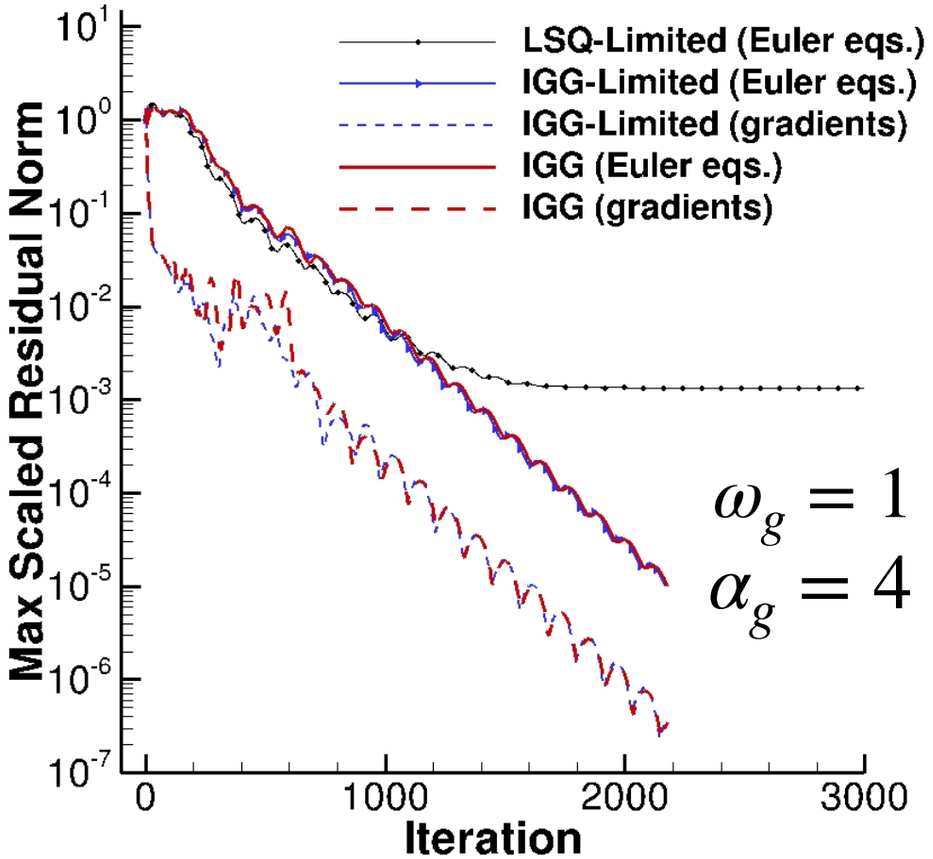


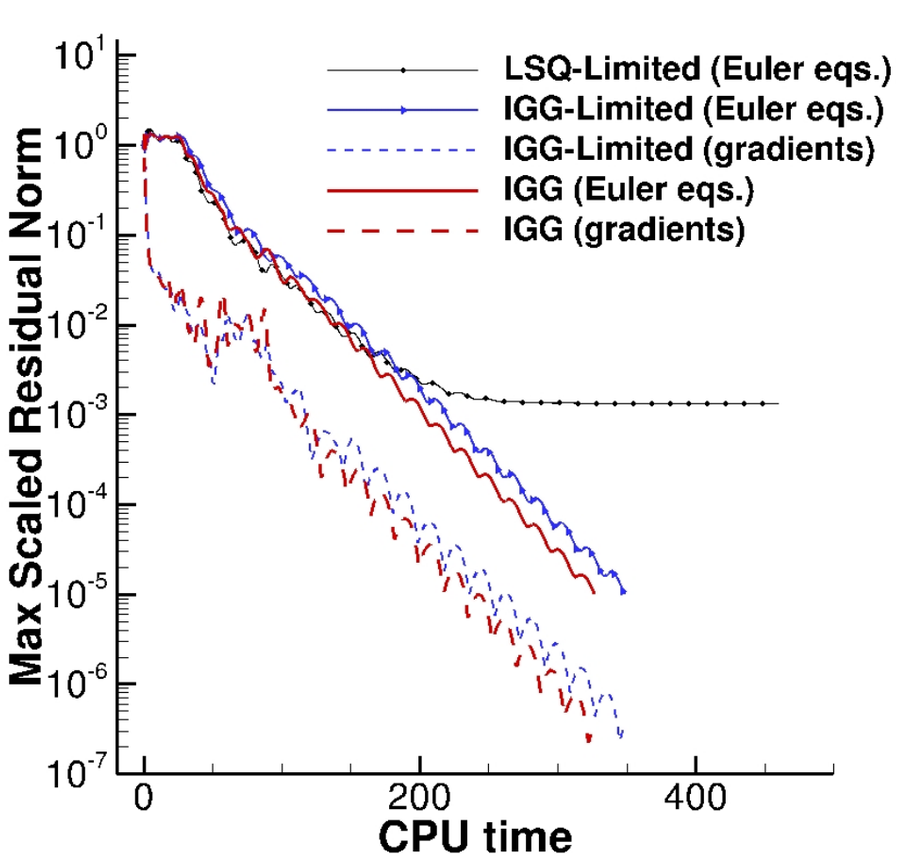






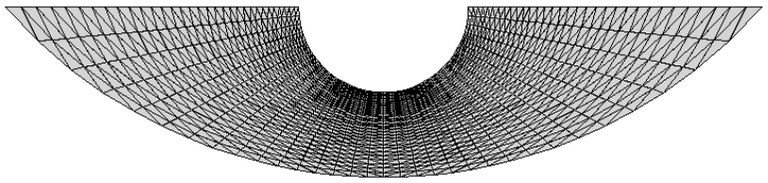
Limiter is Venkat with vertex enforcement (MLP).

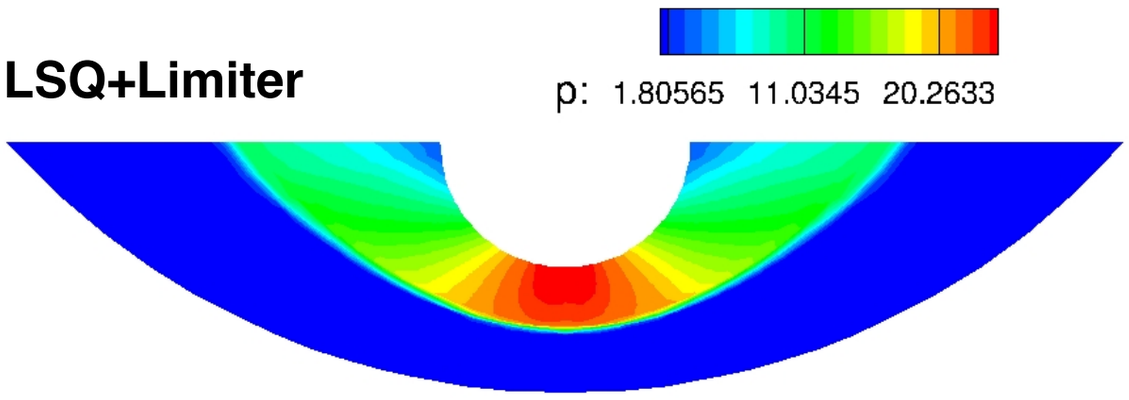


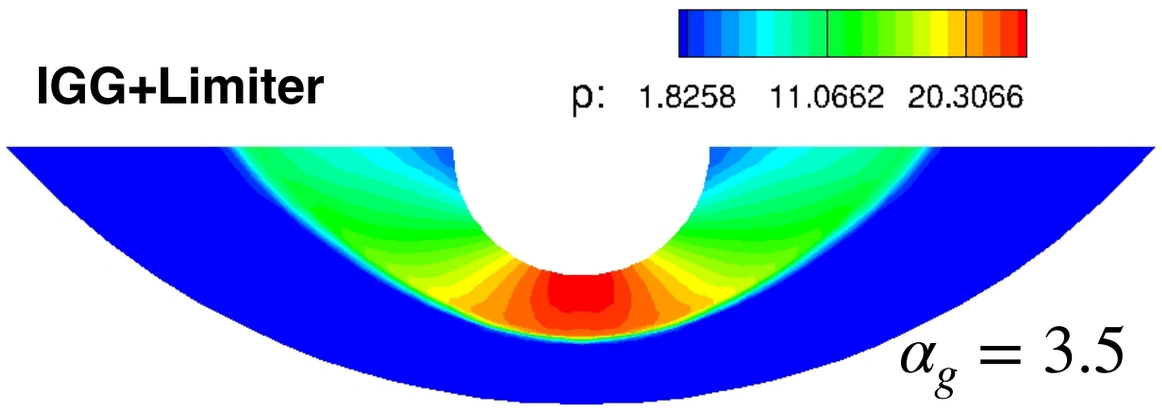


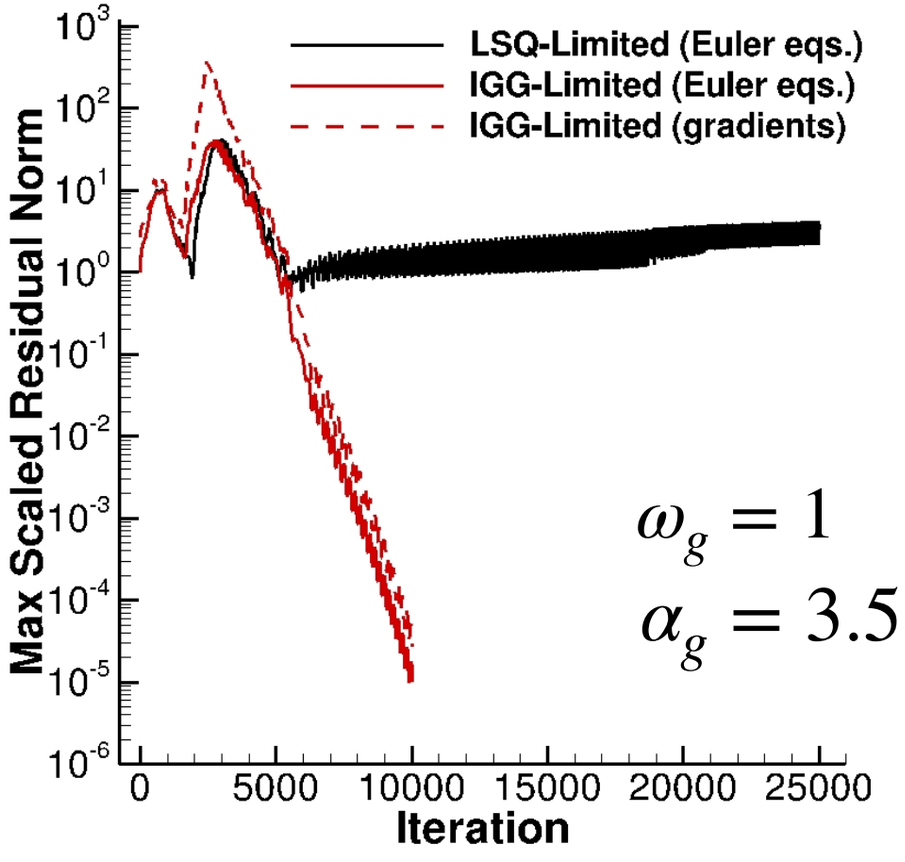
IDC-IGG converges with/without a limiter on highly-irregular grid.

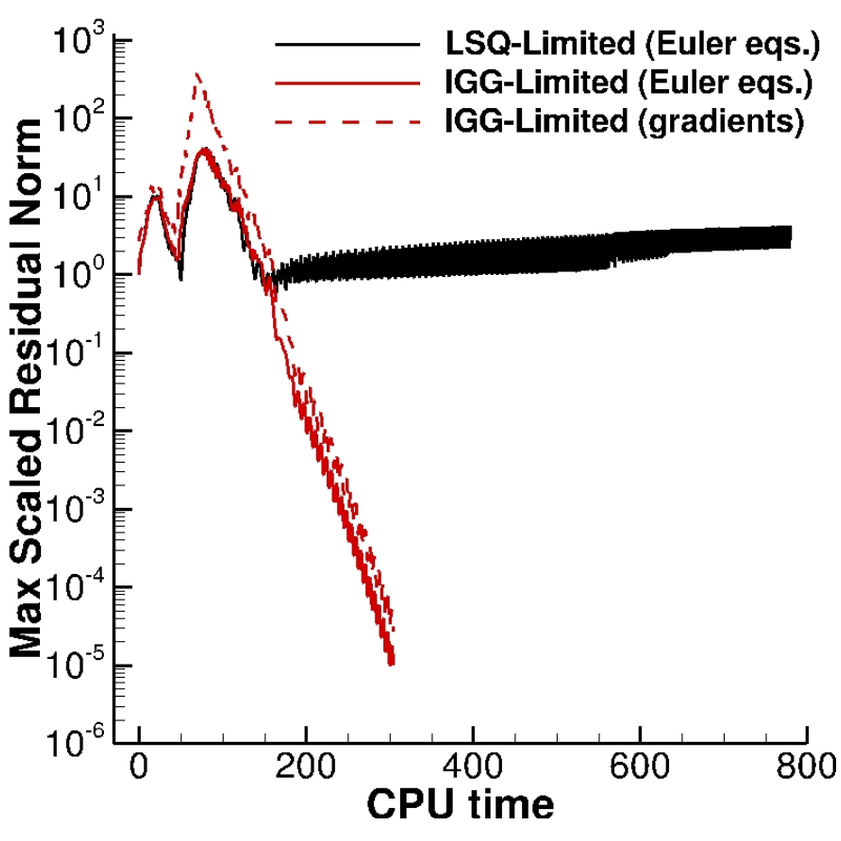
Hypersonic Cylinder at M=5.2 (HLL)











IDC-IGG converges with a large damping coefficient.

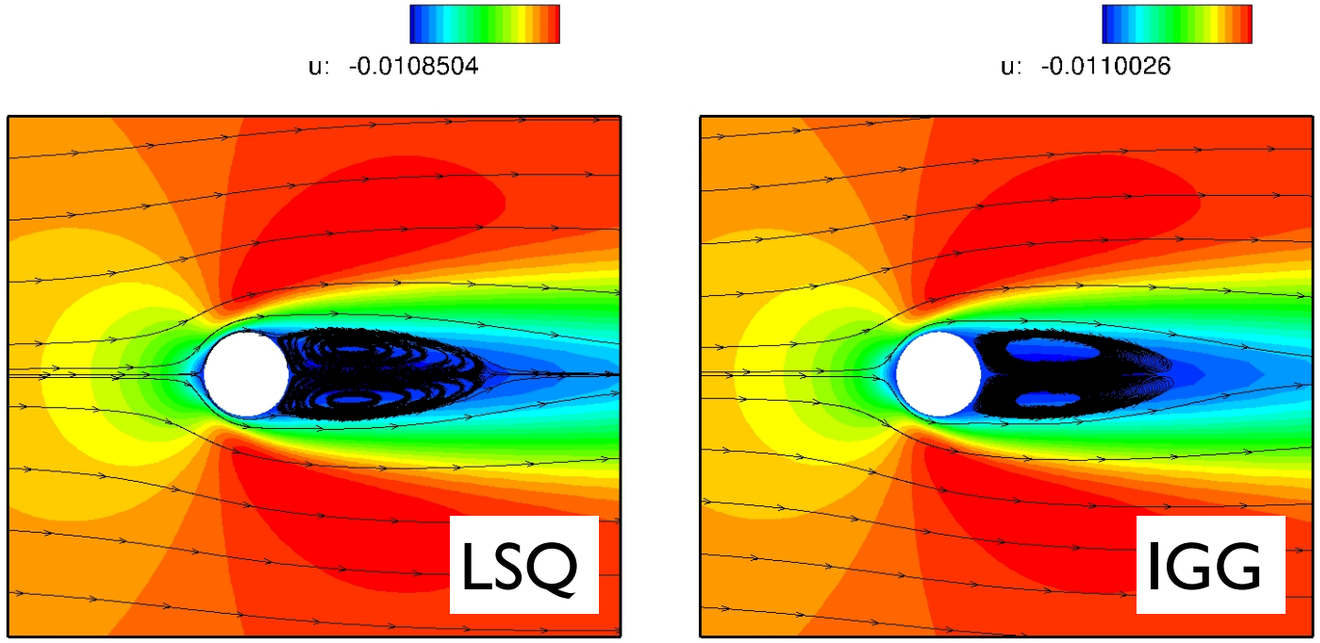
Low-Re Cylinder at Mach=0.3, Re=40 (Roe, Alpha-damp)

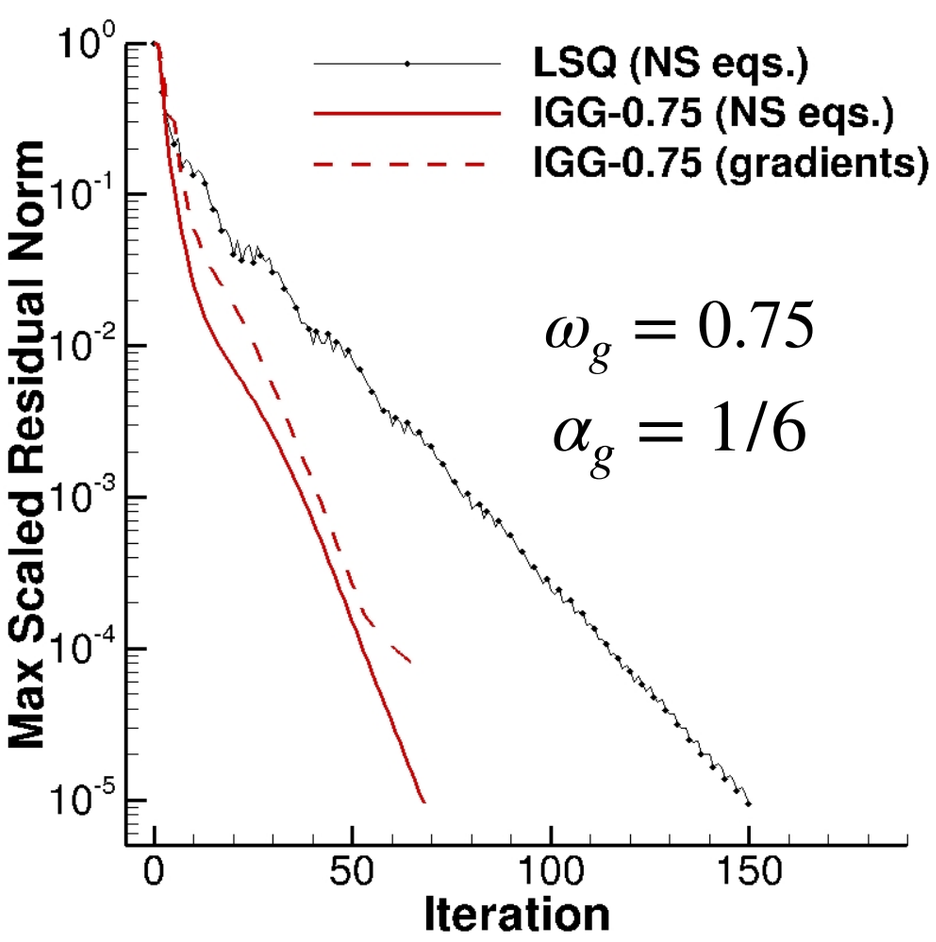
IDC-IGG solver diverged with

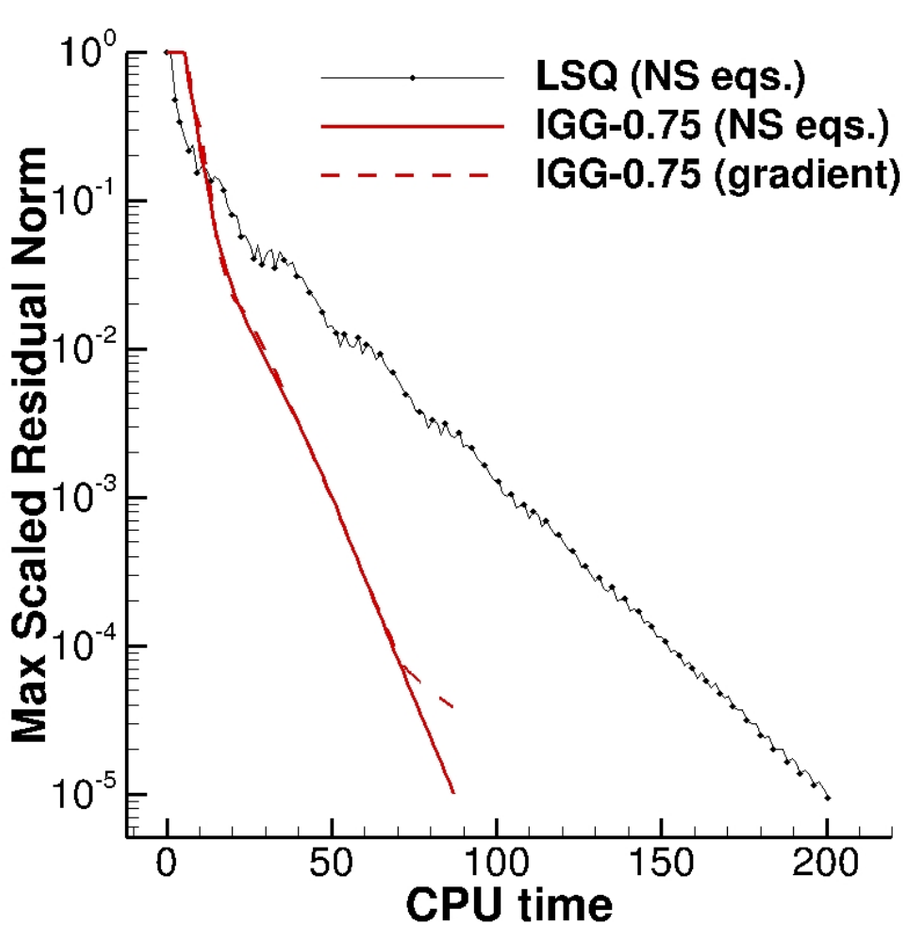
$$\omega_g = 1.0$$

which needed to be reduced to

$$\omega_g = 0.75$$



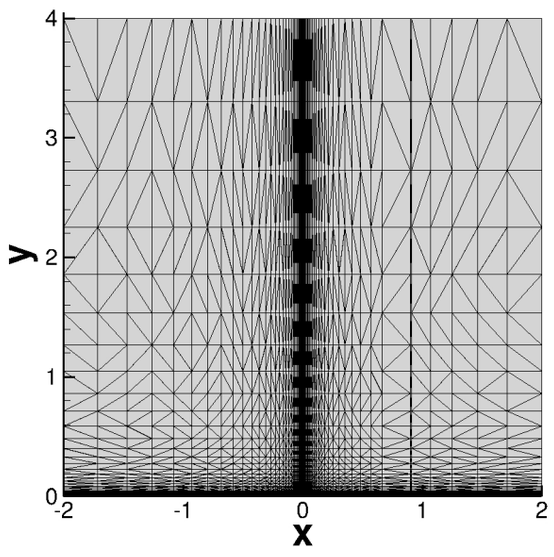


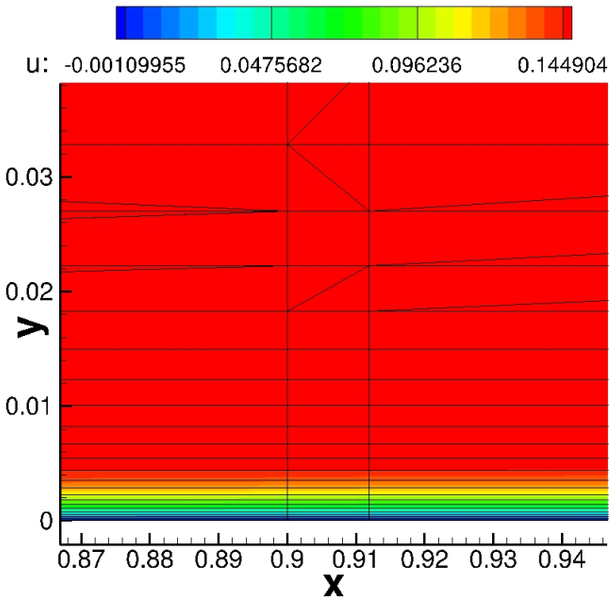


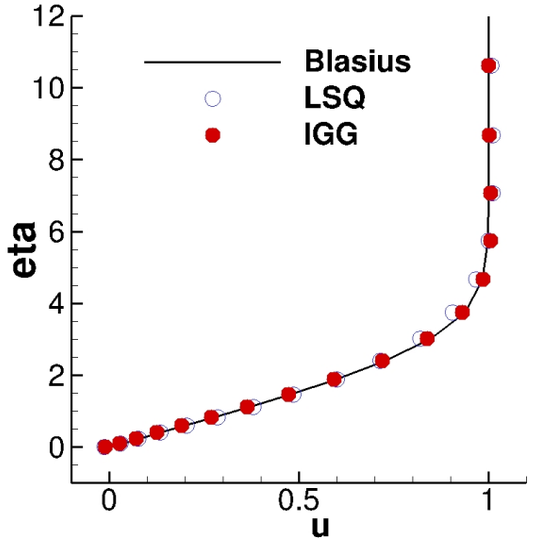
IDC-IGG converges faster than with LSQ.

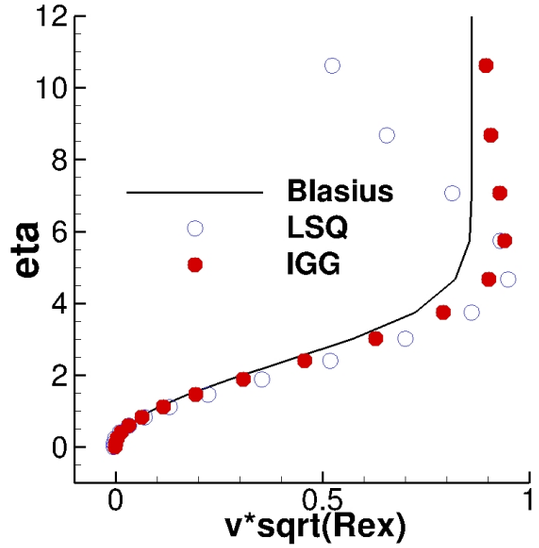
High-Re Flat Plate at M=0.15, Re=10^6 (Roe, Alpha-damp)

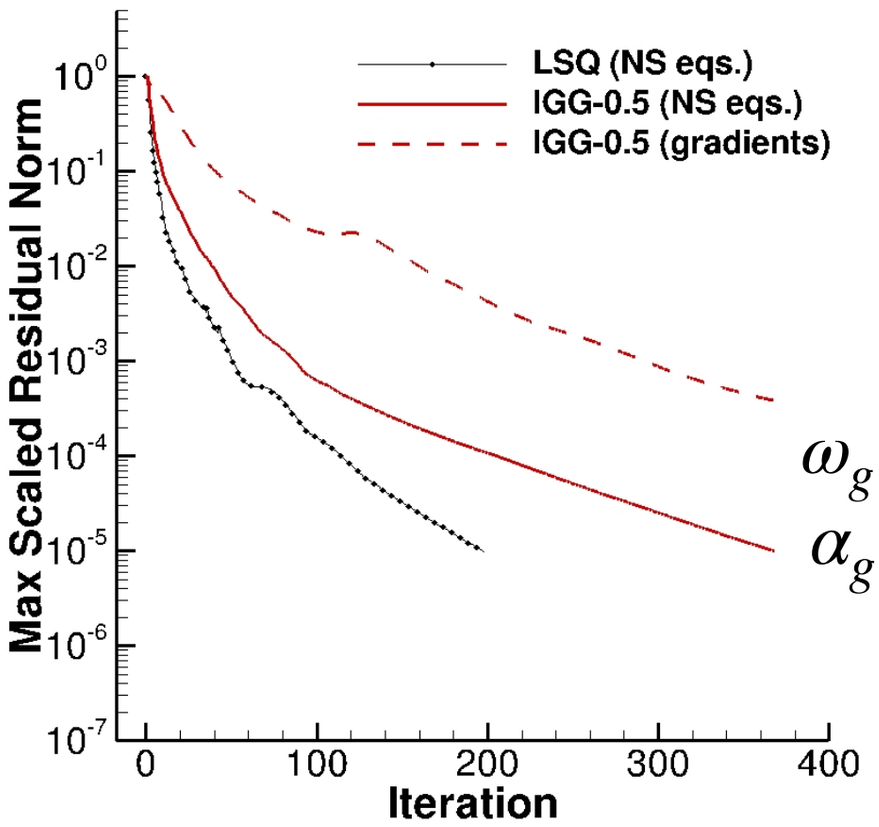
IDC-IGG solver diverged with $\omega_g = 0.75$ which needed to be reduced to $\omega_g = 0.5$





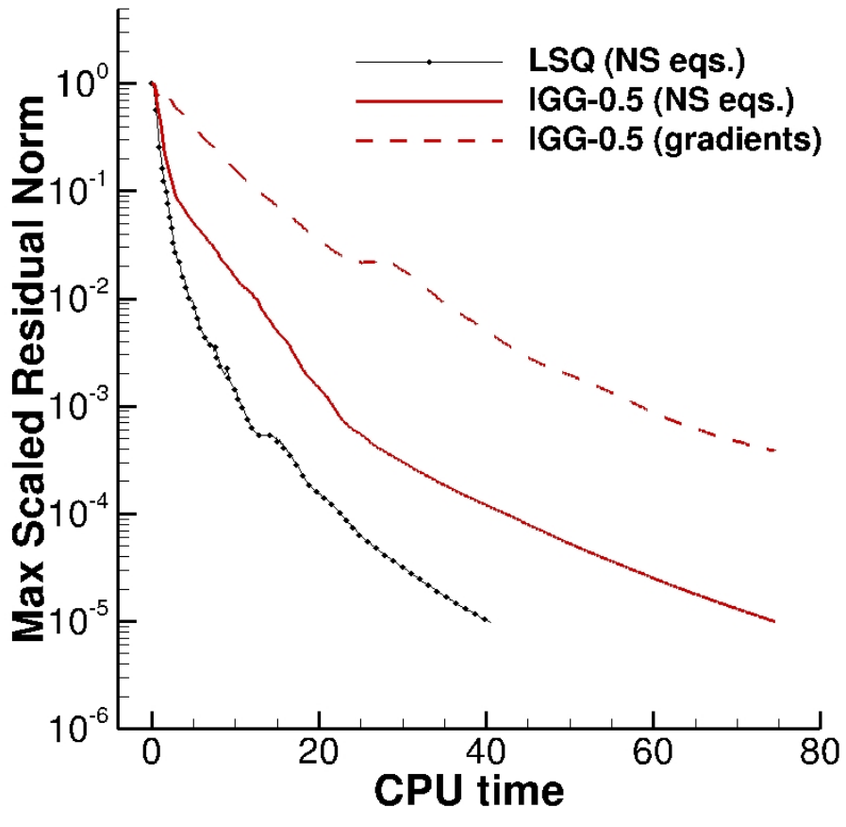




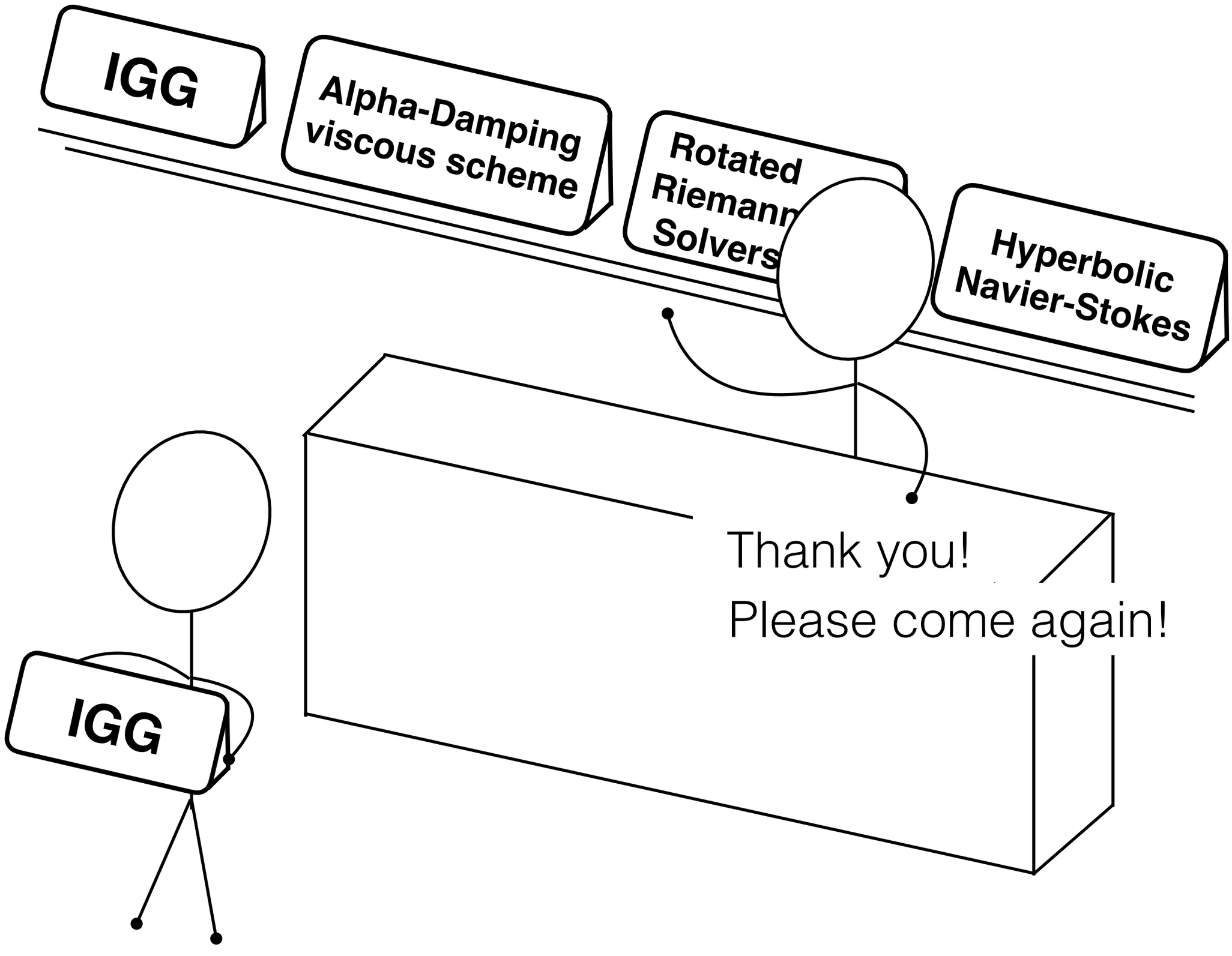


$\omega_g = 0.5$

$\alpha_g = 1/6$



IDC-IGG is slower here but produces more accurate solution.


IGG
Alpha-Damping viscous scheme
Rotated Riemann Solvers
Hyperbolic Navier-Stokes
Thank you!
Please come again!
IGG

Conclusions

National Institute of Aerospace

- IGG gradients demonstrated for subsonic to hypersonic flows.
- Affordable and as cheap as LSQ with "Installment plan": relax per iteration
- IGG gradient can be adjusted so that a solver converges without a limiter.
- It can stabilize and speed up computations.
- Instability can occur, but can be overcome with a small relaxation factor.

Better to gain gradient accuracy slowly?

Future work:

- More tests especially for practical problems in 3D.
- Explore Plan 3: Newton solver
  - * Very robust but many linear relaxations required: Very efficient linear solver needed.
  - * 20 residual and 20x20 Jacobean blocks -> then, HNS is more advantageous?
- Adaptive IGG with a locally defined coefficient: α_g

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