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Third-Order Edge-Based Scheme for Unsteady Problems

Acknowledgements:
NASA LaRC FUN3D group, and
Dr. Mujeeb Malik in the Revolutionary Computational Aerosciences (**RCA**) project
within NASA's Transformational Tools and Technologies (TTT) Project
of the Transformative Aeronautics Concepts Program under the Aeronautics Research Mission Directorate.

Hiro Nishikawa and Yi Liu

48th AIAA Fluid Dynamics Conference, June 29, 2018

Good, Better, and Best

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3 Trim Models!
The higher the price, the higher the quality!

CIVIC LX
$19,000~

CIVIC EX
$21,000~

CIVIC EX-T/L
$24,000~

Which one would you choose?
How do you choose one?

2nd, 3rd, and 4th Order

3 Order Models!
The higher the order,
the higher the quality!

2nd-order
1~4 DoF/var

3rd-order
1~10 DoFs/var

4th-order
1~20 DoFs/var

Which one would you choose?
How do you choose one?

Immediate Improvements

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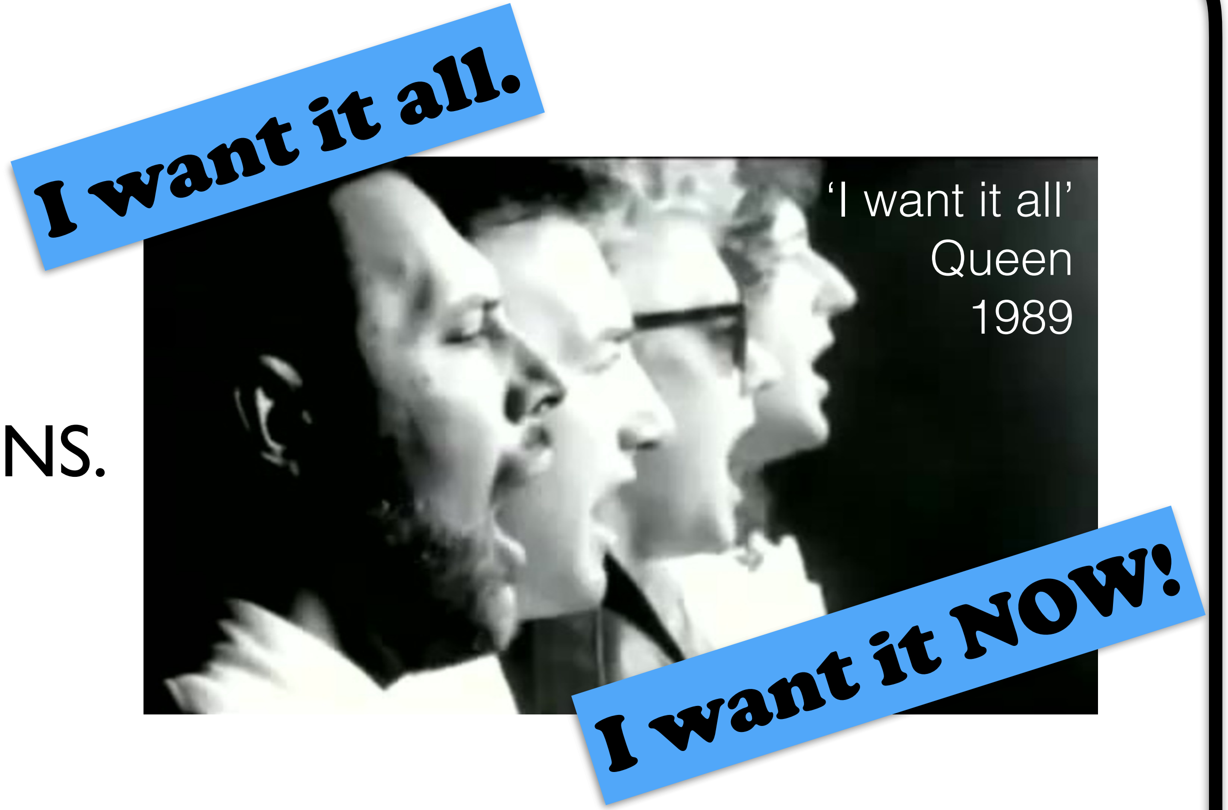
High-order methods are sought for high-fidelity simulation on unstructured grids…

Progress made, but still practical CFD codes rely on low-order methods…

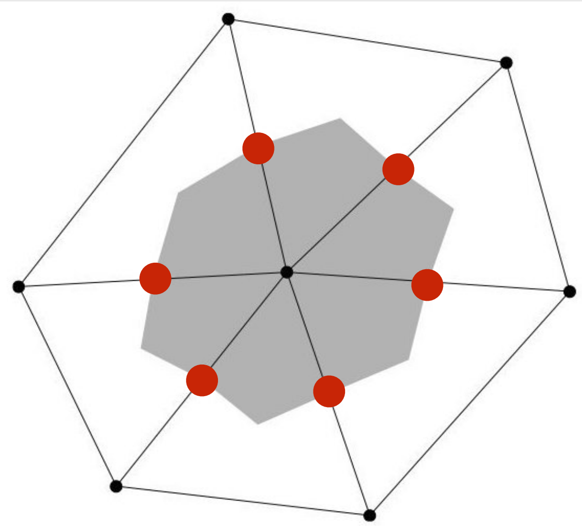
1. Major code restructuring and verification.
2. High computational cost.
   e.g., 50 eqns solved in 3rd-order DG for NS.
3. Grid adaptation for efficiency.
   e.g., target accuracy on coarser grids.
4. High-order curved grid generation and adaptation.



You can have it now by special high-order methods.

Third-Order Edge-Based Scheme

Note: EB scheme is 2nd/3rd-order on simplex grids.



Originally discovered by Katz and Sankaran for 2D conservation laws (2011).

$$\mathrm{div}\,\mathbf{f} = 0 \longrightarrow \frac{1}{V_j}\sum_{k\in\{k_j\}} \phi_{jk}|\mathbf{n}_{jk}| = 0$$

3rd-order by nearly 2nd-order algorithms

- Quadratic LSQ gradients
- Linear flux extrapolation: $\mathbf{f}_L = \mathbf{f}_j + \frac{1}{2}\left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right)_j \nabla \mathbf{u}_j \cdot \Delta \mathbf{r}_{jk}$
- Special source quadrature (e.g., unsteady terms)

- 3rd-order on linear tetrahedral grids for curved geometries (JCP2016)
- 3rd-order without second-derivatives (Theory in JCP2017: NIA Best Paper 2017)
- 3rd-order on zero/negative-volume grids (AIAA Best CFD Paper 2017)

3D Navier-Stokes version is implemented in NASA's FUN3D.

AIAA2016-2969, AIAA2017-0081, AIAA2017-0738 (Unsteady), AIAA2017-3443

Model Unsteady Problems

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Consider 1D advection-diffusion equation:

$$\partial_t \mathbf{u} + \partial_x \mathbf{f} = \mathbf{s}$$

Implicit time-integration:

$$\partial_x \mathbf{f} = \mathbf{s} - \Delta_t \mathbf{u}$$ <- discretized in time

L-stable BDF2 and **ESDIRK3** schemes:

Kennedy&Carpenter(2003)

[Unconditionally stable] + [Non-oscillatory]

Great to have L-stability for complex applications.

Discretization

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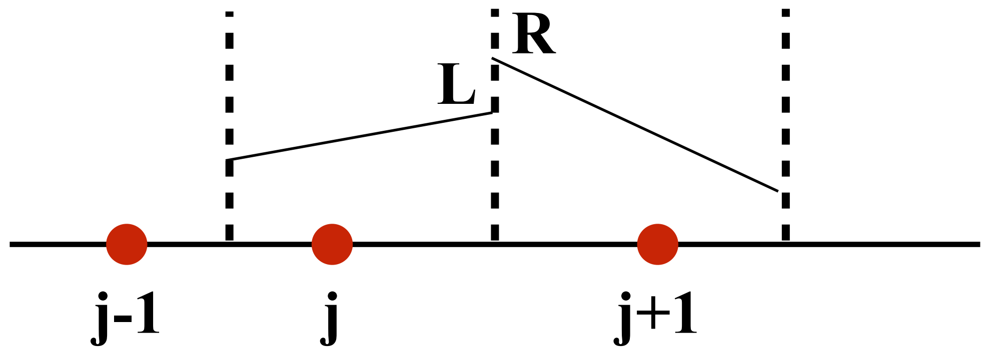
Edge-based discretization:

$$\mathrm{Res}_j = \mathrm{Res}_j^{\Delta x} - \mathrm{Res}_j^{\Delta t}$$



Spatial term:

$$\mathrm{Res}_j^{\Delta x} = -\left(\frac{\Phi_{j+1/2} - \Phi_{j-1/2}}{\Delta x_j}\right) + \frac{1}{\Delta x_j}\int_{\Delta x_j} \mathbf{s}\, dx,$$

$$\Phi(\mathbf{u}_L, \mathbf{u}_R) = \frac{1}{2}\left(\mathbf{f}_L + \mathbf{f}_R\right) - \frac{1}{2}\mathbf{Q}\left(\mathbf{u}_R - \mathbf{u}_L\right)$$

Time term:

$$\mathrm{Res}_j^{\Delta t} = \frac{1}{\Delta x_j}\int_{\Delta x_j} \Delta_t \mathbf{u}\, dx$$

Special quadrature

Nishikawa Liu JCP2017

Time integration schemes

BDF2:
2nd-order

$$\mathbf{Res}_j^{\Delta t} = \frac{1}{\Delta x_j} \int_{\Delta x_j} \left\{ \alpha_p \mathbf{u}^{n+1} + \alpha_n \mathbf{u}^n + \alpha_{n-1} \mathbf{u}^{n-1} \right\} dx$$

$$\boxed{\mathbf{Res}(\mathbf{u}^{n+1}) = \mathbf{Res}^{\Delta x}(\mathbf{u}^{n+1}) - \mathbf{Res}^{\Delta t}(\mathbf{u}^{n+1}, \mathbf{u}^n, \mathbf{u}^{n-1}) = 0}$$

ESDIRK3:
3rd-order

$$\mathbf{Res}_j^{\Delta t}(\mathbf{v}, \mathbf{u}^n) = \frac{1}{\Delta x_j} \int_{\Delta x_j} \frac{\mathbf{v} - \mathbf{u}^n}{\Delta t} dx, \quad \mathbf{v} = \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}^{n+1}$$

$$\boxed{\begin{aligned} \mathbf{u}_1 &= \mathbf{u}^n, \\ \mathbf{Res}^{\Delta t}(\mathbf{u}_2, \mathbf{u}^n) &= \sum_{k=1}^{2} a_{2k} \mathbf{Res}^{\Delta x}(\mathbf{u}_k), \\ \mathbf{Res}^{\Delta t}(\mathbf{u}_3, \mathbf{u}^n) &= \sum_{k=1}^{3} a_{3k} \mathbf{Res}^{\Delta x}(\mathbf{u}_k), \\ \mathbf{Res}^{\Delta t}(\mathbf{u}^{n+1}, \mathbf{u}^n) &= \sum_{k=1}^{4} a_{4k} \mathbf{Res}^{\Delta x}(\mathbf{u}_k). \end{aligned}}$$

Implicit solver is used to solve the unsteady residual eqs.

Nature of Errors

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2nd-order: Dispersive error dominates.

$$T.E.(2nd) = Ch^2 \underline{\partial_{xxx} u}$$

Note: 2nd-order time-integration scheme generates the dispersive error just like a spatial scheme: e.g., for $\partial_t u + a\partial_x u = 0$

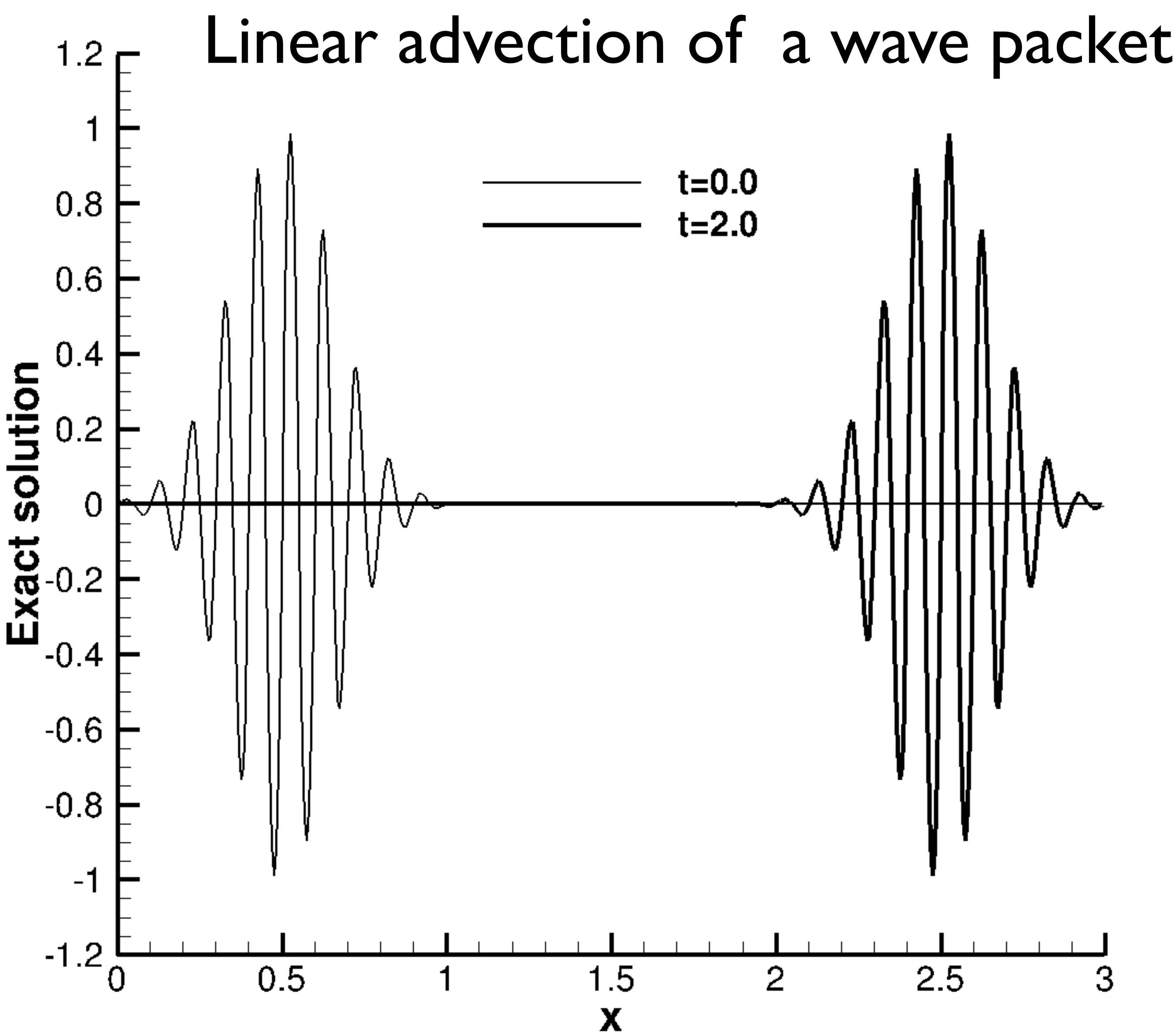
$$T.E.(2nd) = C\Delta t^2 \partial_{ttt} u = C\Delta t^2 a^3 \partial_{xxx} u$$

3rd-order: Dispersive error eliminated.

$$T.E.(3rd) = \cancel{Ch^2 \partial_{xxx} u} + \underline{C_3 h^3 \partial_{xxxx} u}$$

3rd-order solution should be smoother without large oscillations and propagate at a correct speed.

Test Problem



Nature of Errors

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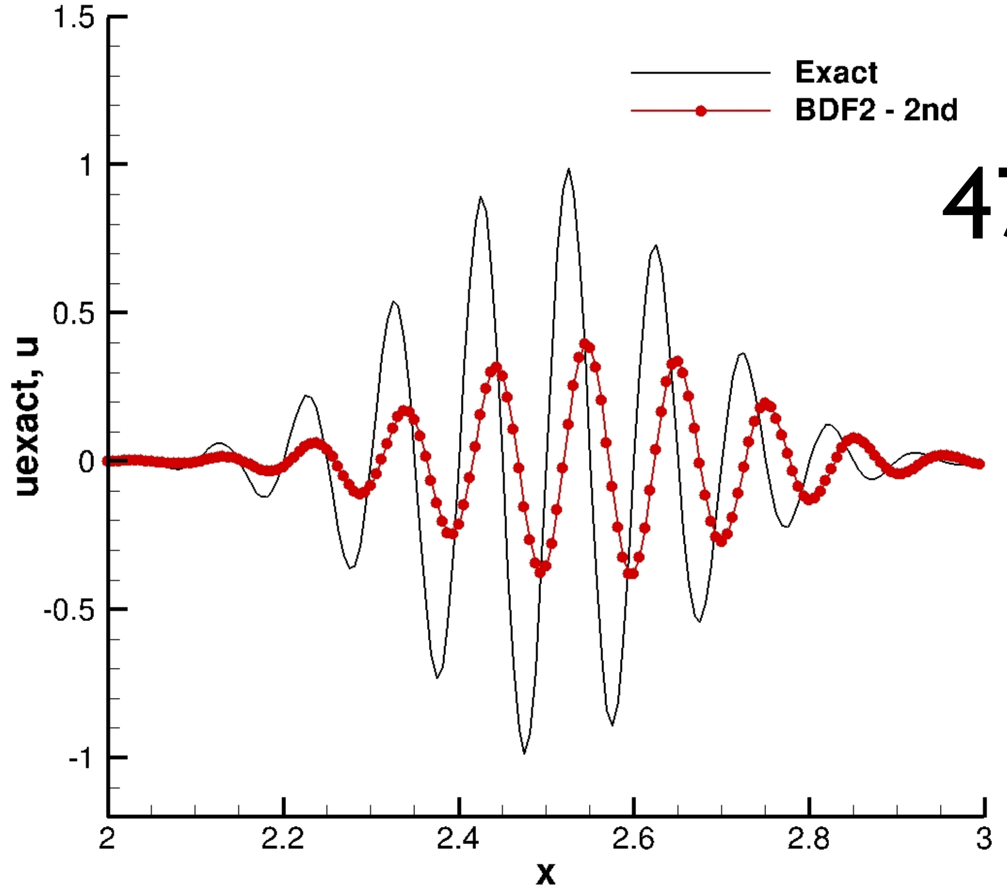
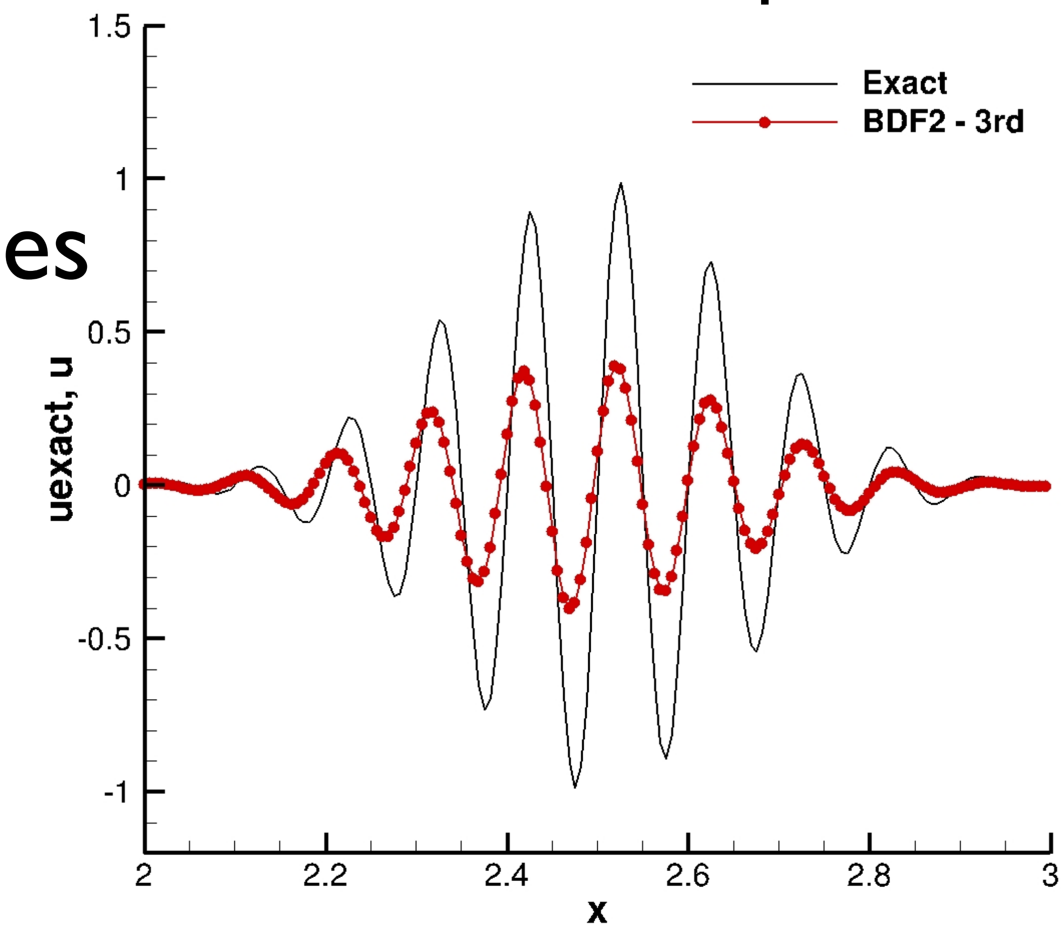
2nd-order time-integration scheme: BDF2 at dt=0.001

2nd-order in space



479 nodes

3rd-order in space



Spatial error dominates: 3rd-order scheme is more accurate even with BDF2.

"Good, but I want less dissipative solution….."

Then, how about 4th-order?

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2nd-order: Dispersive error dominates.

$$T.E.(2nd) = C h^2 \underline{\partial_{xxx} u}$$

3rd-order: Dispersive error eliminated.

$$T.E.(3rd) = \cancel{C h^2 \partial_{xxx} u} + \underline{C_3 h^3 \partial_{xxxx} u}$$

4th-order: Dispersive and dissipative errors eliminated.

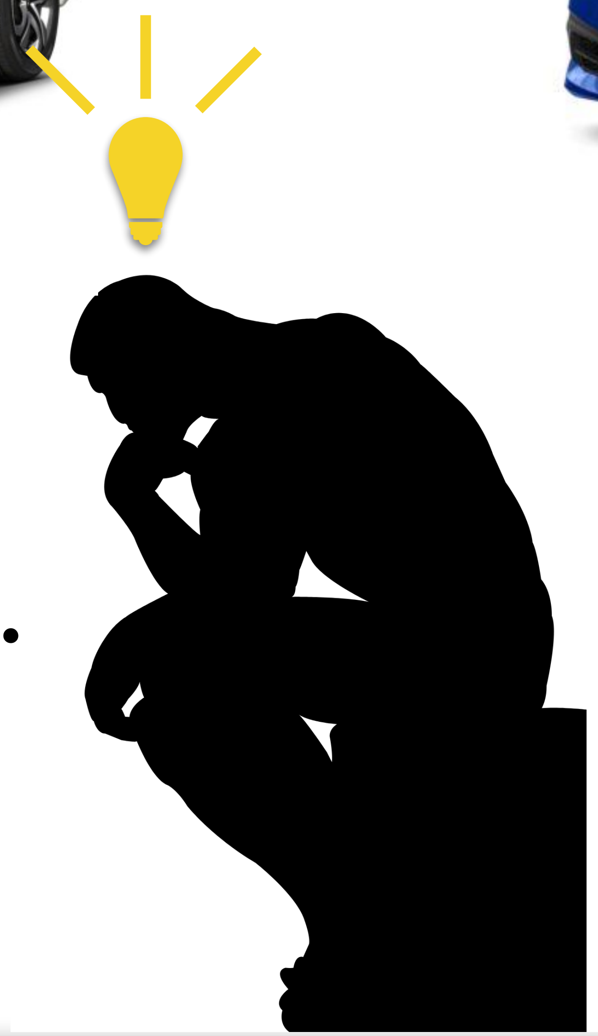
$$T.E.(4th) = \cancel{C h^2 \partial_{xxx} u} + \cancel{C_3 h^3 \partial_{xxxx} u} + \underline{C_4 h^4 \partial_{xxxxx} u}$$

OK, but *can you afford?*

"I want heated seats…"

CIVIC EX
$21,000~

No heated seats…



CIVIC EX-T/L
$24,000~

Heated seats!!

Get It cheaper!

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CIVIC EX

$21,000~

+

Cheap Heating Blanket

<$100

VS

CIVIC EX-T/L

$24,000~

Heated seats

Can I get low dissipation cheaper?

3rd-order

1~10 DoFs/var

+

$$\frac{1}{2}\left(\mathbf{f}_L + \mathbf{f}_R\right) - \frac{1}{2}\underbrace{\mathbf{Q}\left(\mathbf{u}_R - \mathbf{u}_L\right)}_{\text{Dissipation}}$$

Reduce (uR-uL) by high-order reconstruction?

$$\mathbf{u}_L = \mathbf{u}_j + \frac{1}{2}\partial_{jk}\mathbf{u}_j + \frac{1}{8}\partial_{jk}^2\mathbf{u}_j \ \ldots$$

High-order derivatives are required……
kappa=1/2 does quadratic, but no significant impact…

VS

4th-order

1~20 DoFs/var

Low dissipation

Get low dissipation cheaper!

3rd-order

1~10 DoFs/var

+

Reduce the coefficient Q, instead of the jump (uR-uL) !!!

Low dissipation

$$\Phi = \frac{1}{2}\left(\mathbf{f}_L + \mathbf{f}_R\right) - \frac{1}{2}\mathbf{Q}^*\left(\mathbf{u}_R - \mathbf{u}_L\right)$$

$$\mathbf{Q}^* = K\mathbf{Q} \quad K = 0.01$$

Almost no add. cost !

VS

4th-order

1~20 DoFs/var

Low dissipation

ESDIRK3 with a large dt

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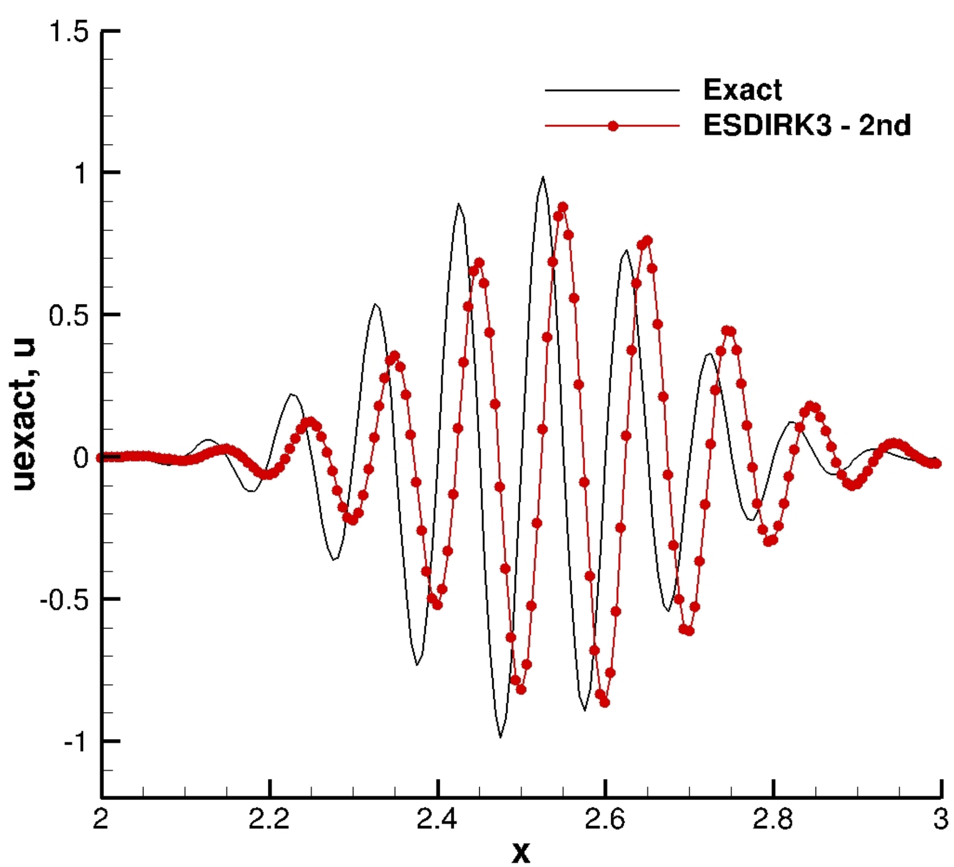
ESDIRK3 and low-dissipation flux at dt=0.005

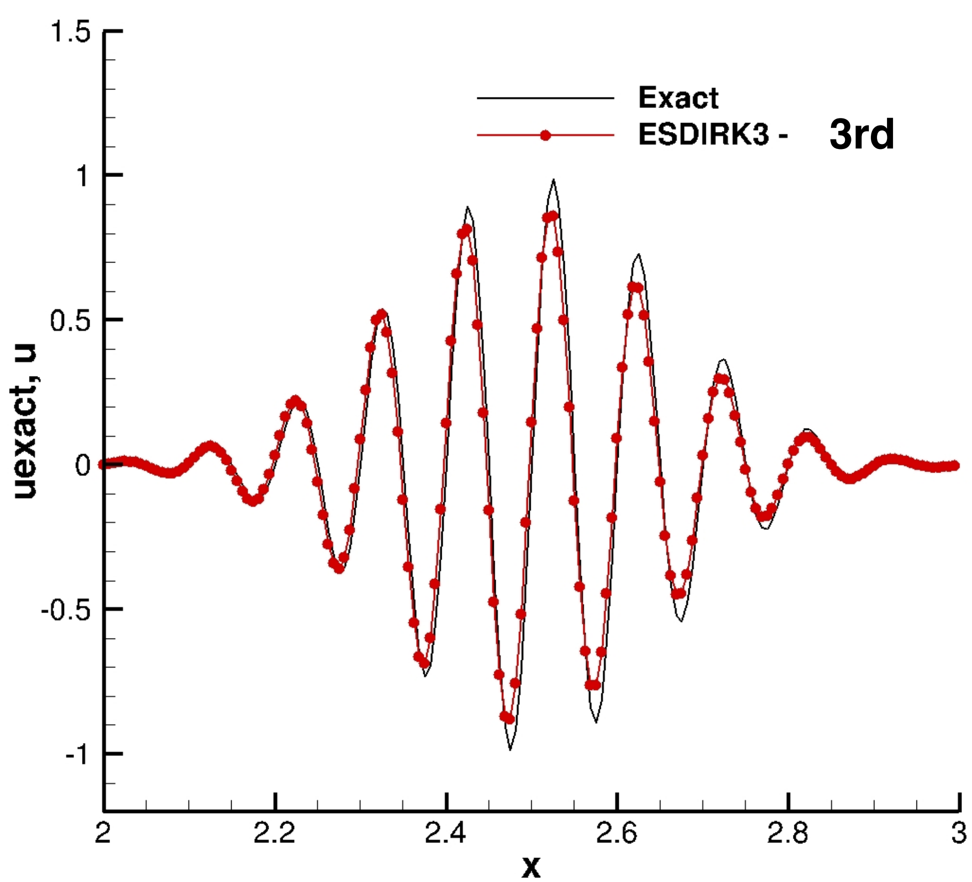
3rd-order in time + 2nd-order in space



Dispersive…

Waste of resources…..

3rd-order in space and time

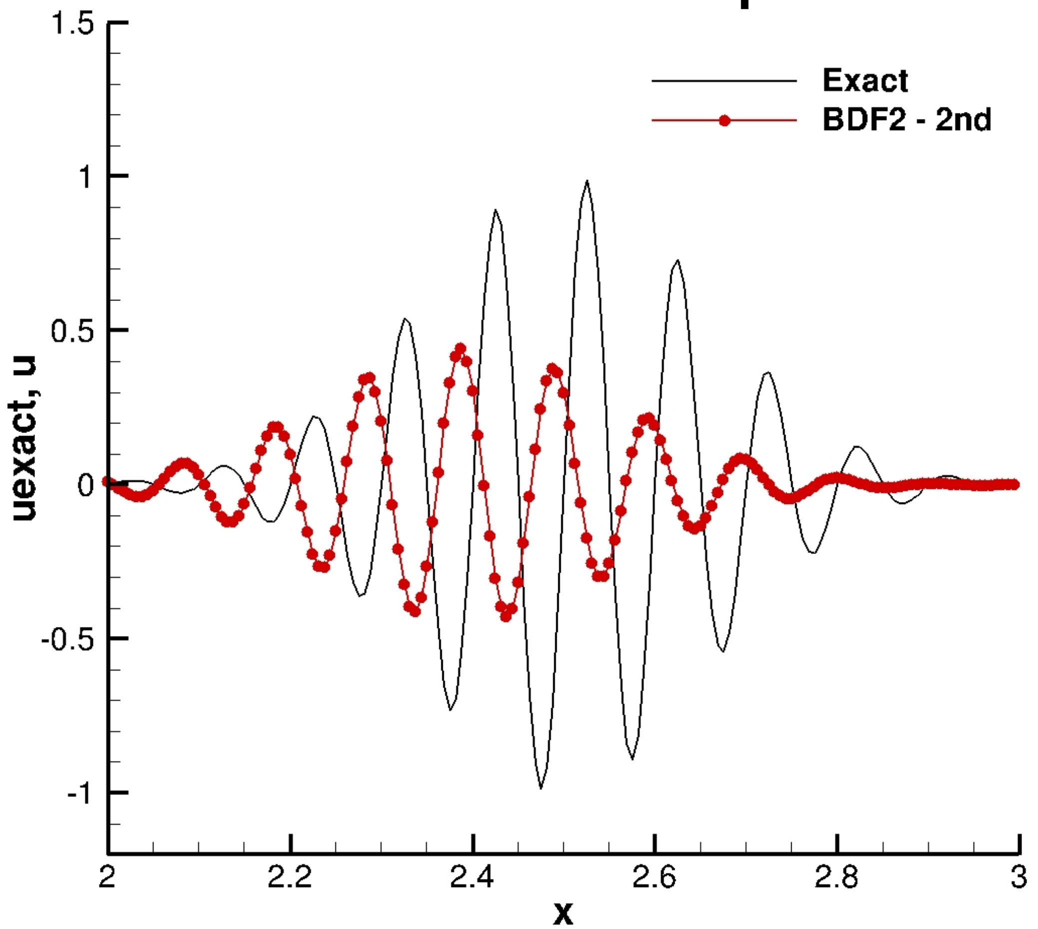


Excellent solution

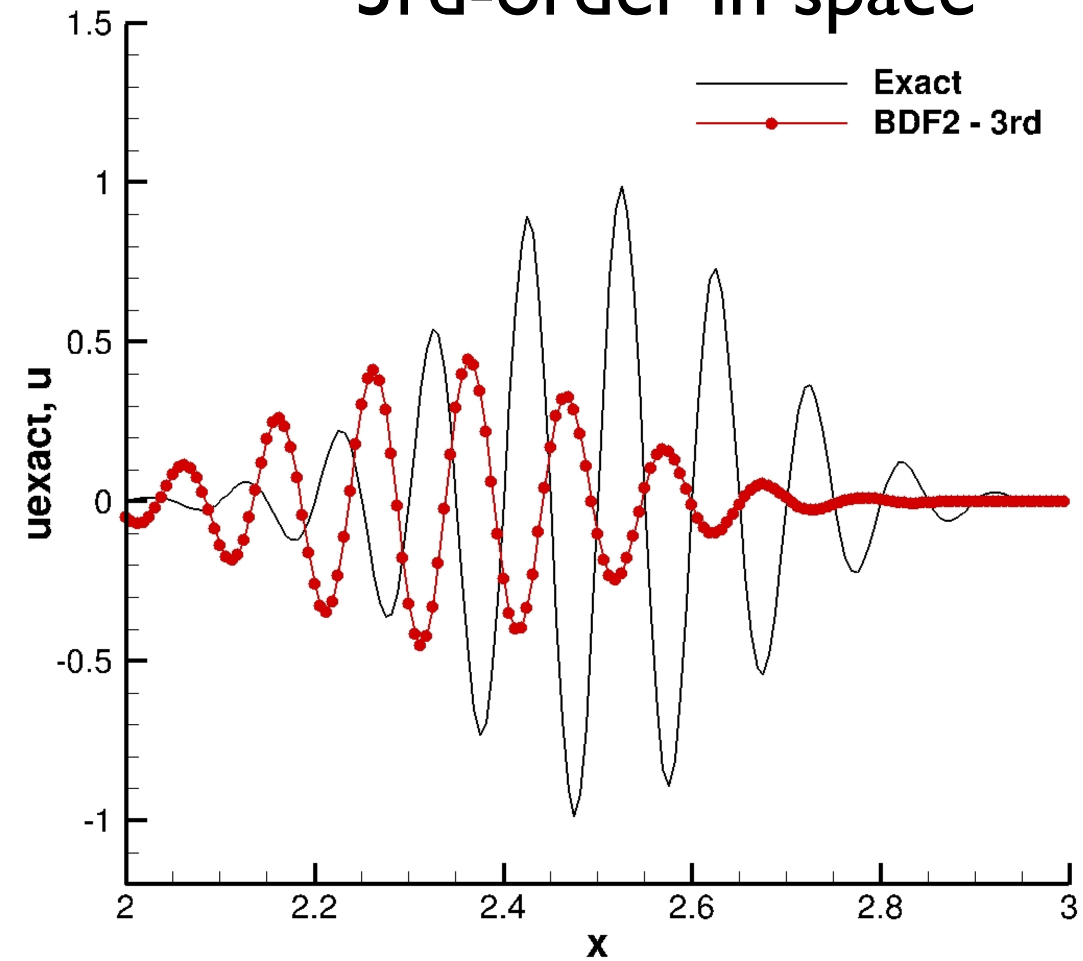
BDF2 with a large dt

BDF2 and low-dissipation flux at dt=0.005

2nd-order in space



3rd-order in space



Waste of resources…..

BDF2 error dominates…

No advantage by 3rd-order nor low-dissipation…..

Irregular grid at dt = 0.025

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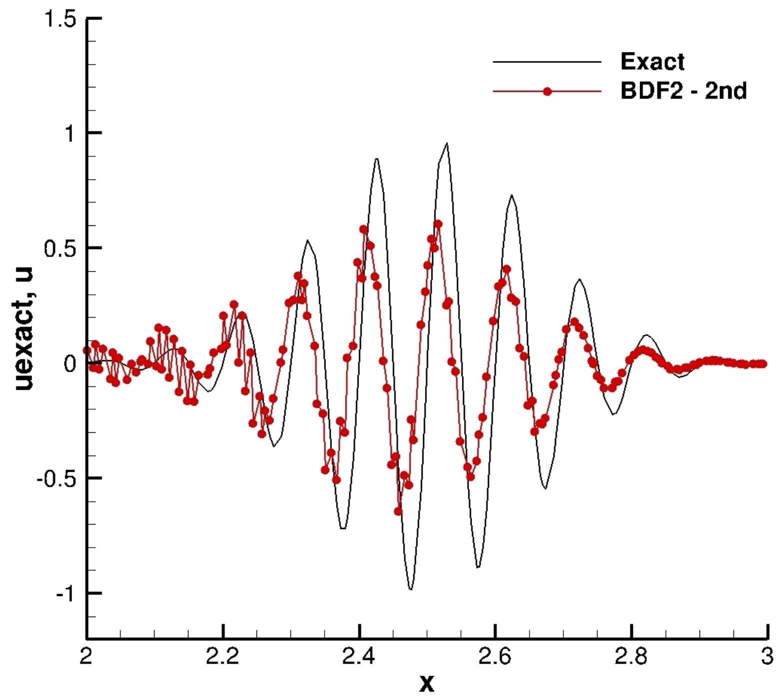
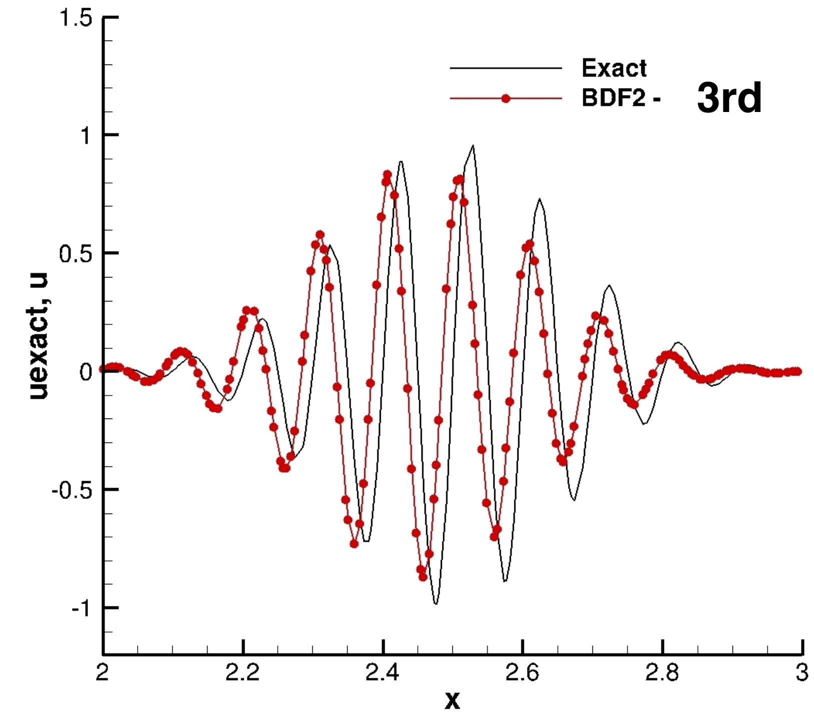
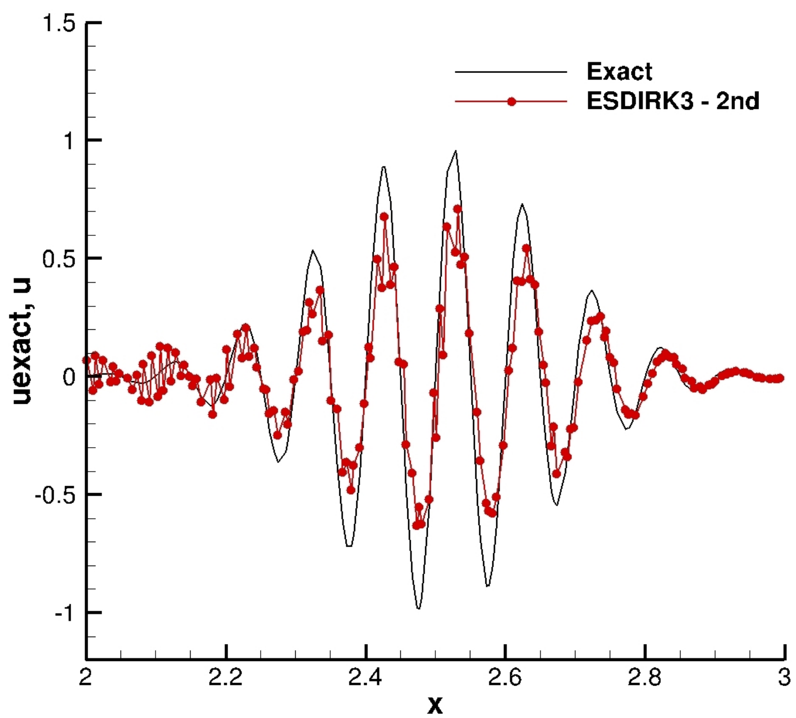
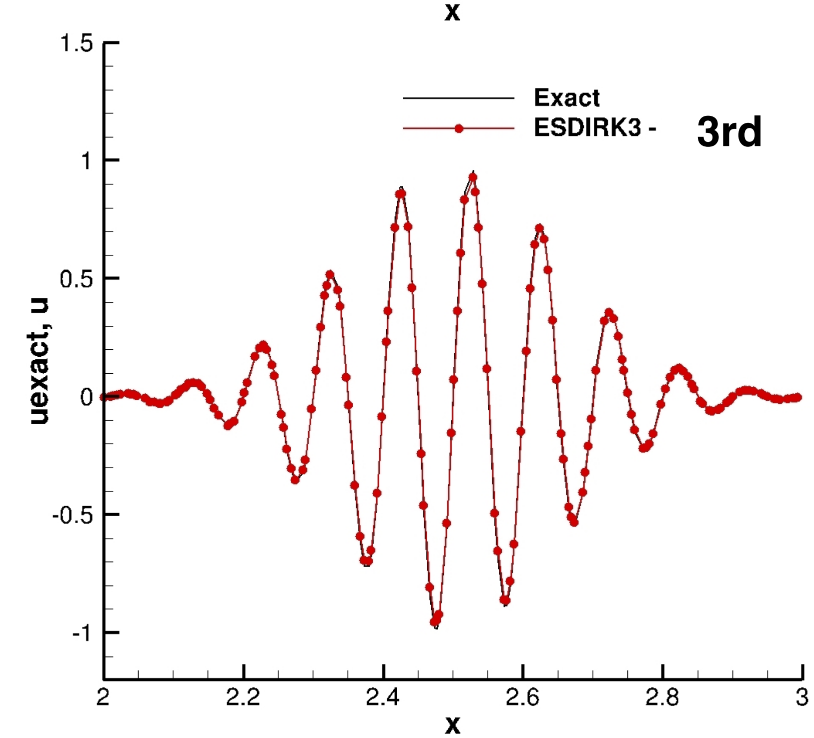
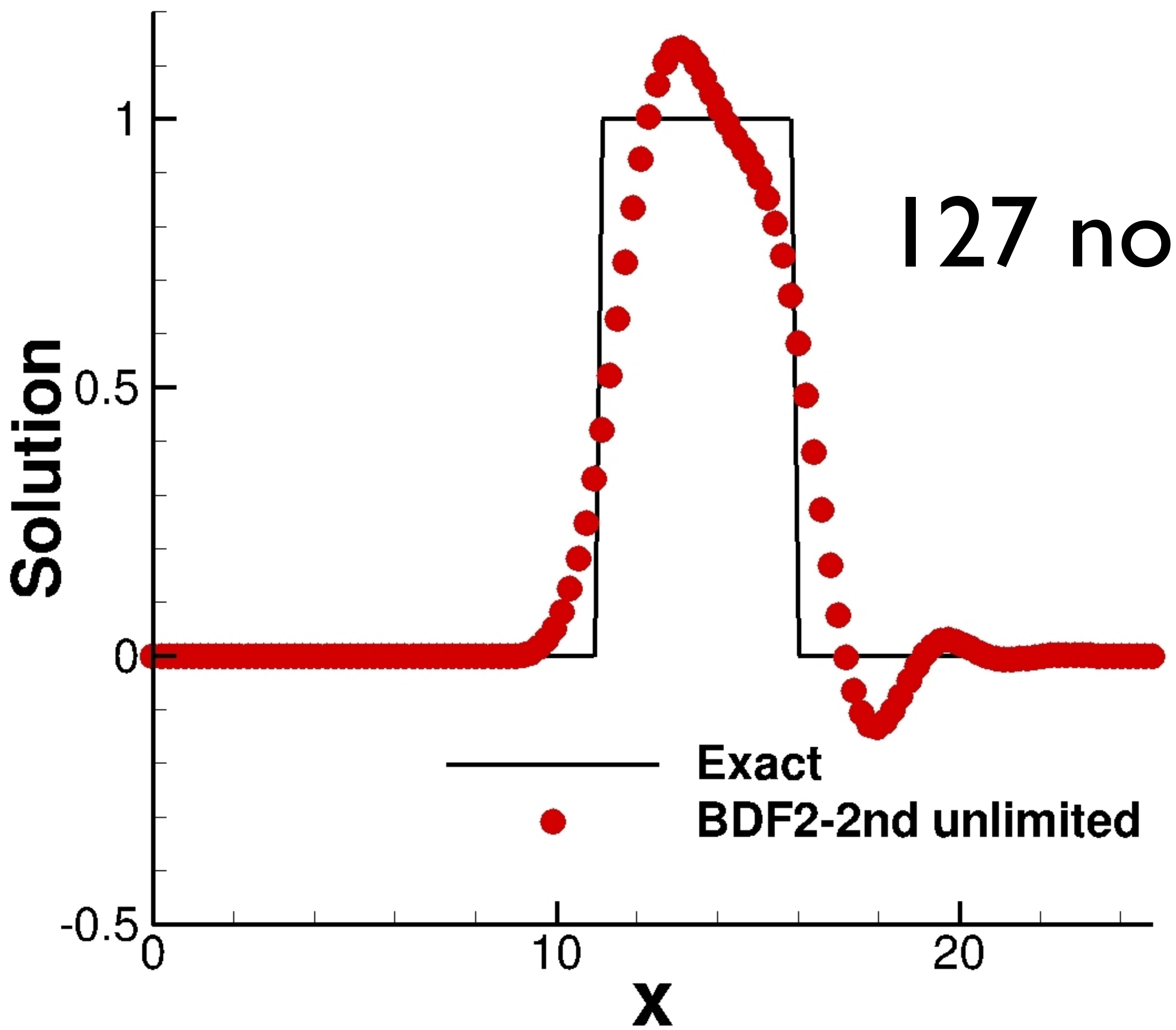
Time \ Space	2nd	3rd
2nd		
3rd		

Convection of a square profile

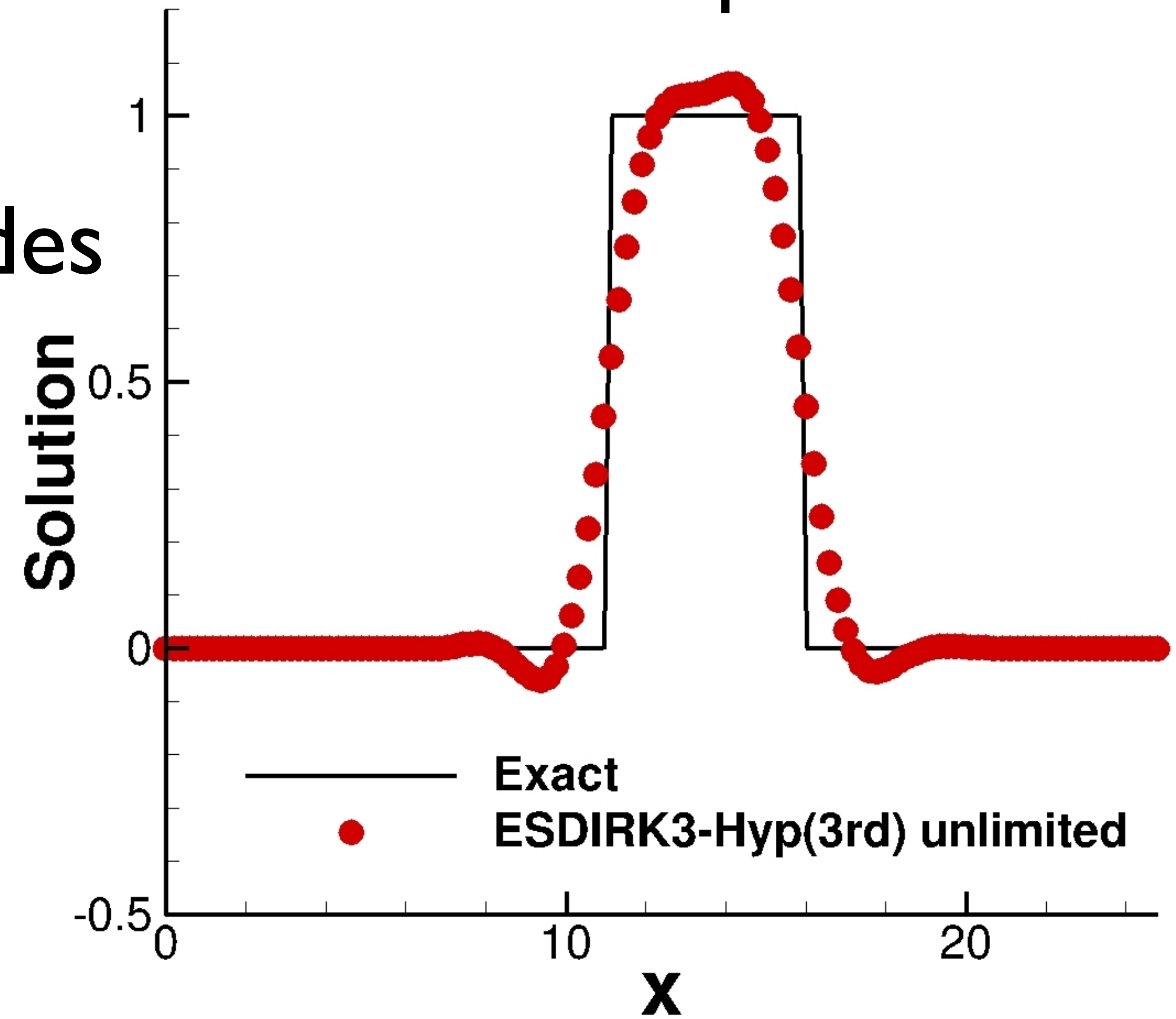
Convection over distance 60 with dt=0.2

2nd-order in space and time



127 nodes

3rd-order in space and time



3rd-order is low-dispersive and accurate with no limiter.

Low-Dissipation Roe Flux

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Roe flux: $$\frac{1}{2}(F_{nL} + F_{nR}) - \frac{1}{2}|\mathbf{A}_n|(\mathbf{U}_R - \mathbf{U}_L)$$

Low-Dissipation Roe: Rieper(2011), Thornber et.al.(2008), Oβwald et.al.(2016).
Compute the averaged flux, and then modify the velocities

$$\mathbf{v}_L^* = \frac{\mathbf{v}_L + \mathbf{v}_R}{2} - \frac{z}{2}(\mathbf{v}_R - \mathbf{v}_L) \qquad z = \min(1, \max(M_L, M_R))$$

$$\mathbf{v}_R^* = \frac{\mathbf{v}_L + \mathbf{v}_R}{2} + \frac{z}{2}(\mathbf{v}_R - \mathbf{v}_L)$$ Thornber et.al.(2008)

to compute the whole dissipation term $|\mathbf{A}_n|(\mathbf{U}_R - \mathbf{U}_L)$

Effects:

Low-Mach fix of Rieper: $\Delta u_n^* = z\Delta u_n$

Reduced dissipation (L2Roe): $\Delta \mathbf{v}^* = z\Delta \mathbf{v}$ Oβwald et.al.(2016)

NASA's FUN3D

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Released: Very economical high-resolution scheme

FUN3D-**i3rd**: 3rd-order inviscid scheme

FUN3D-**LDRoe**: flux_construction = "ldroe"

Not yet released: exists in a branch

FUN3D-**HNS**:

- 3rd-order Navier-Stokes scheme
- 3rd-order for both flow variable and their gradients
- Convergence acceleration on refined grids

Laminar flow over a cylinder

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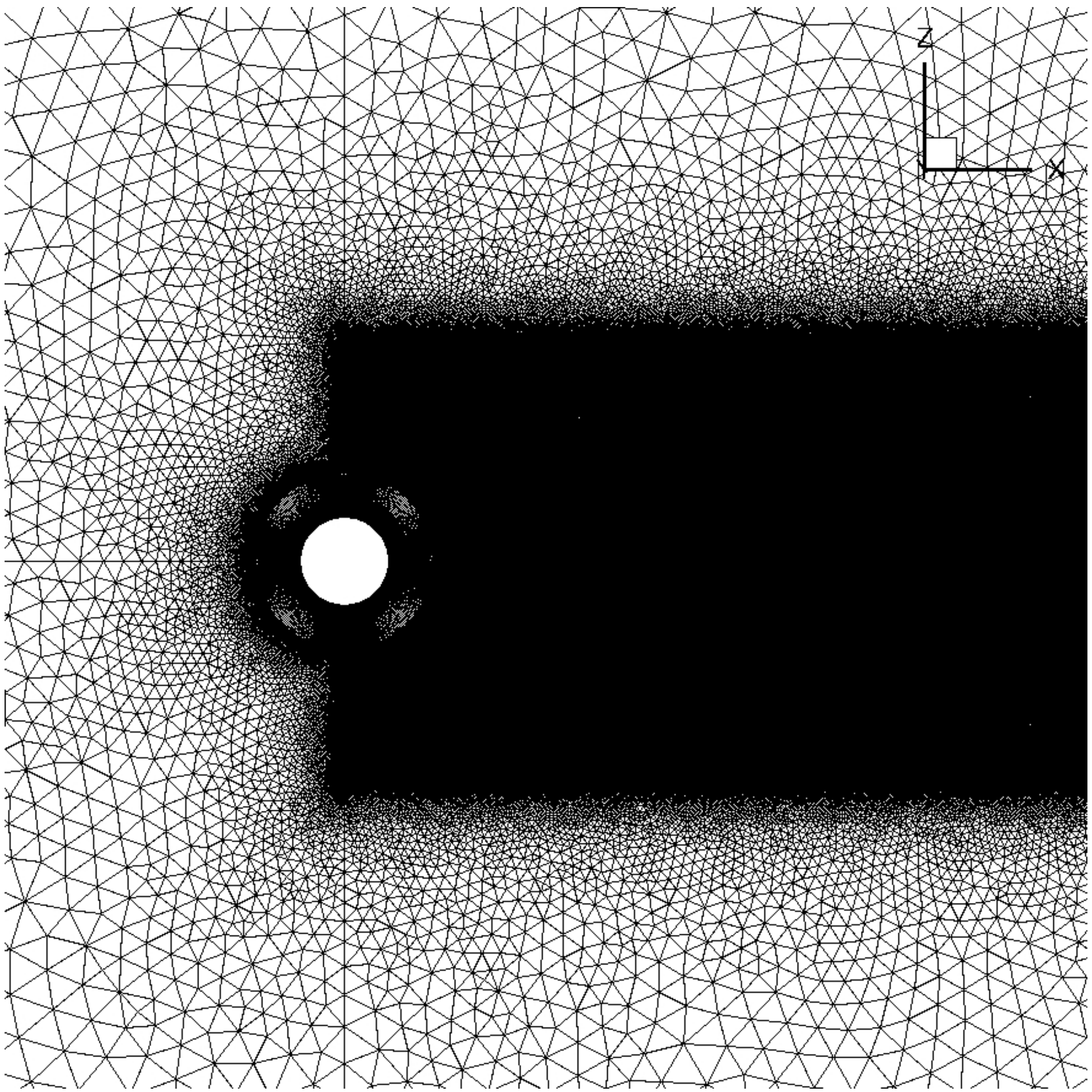
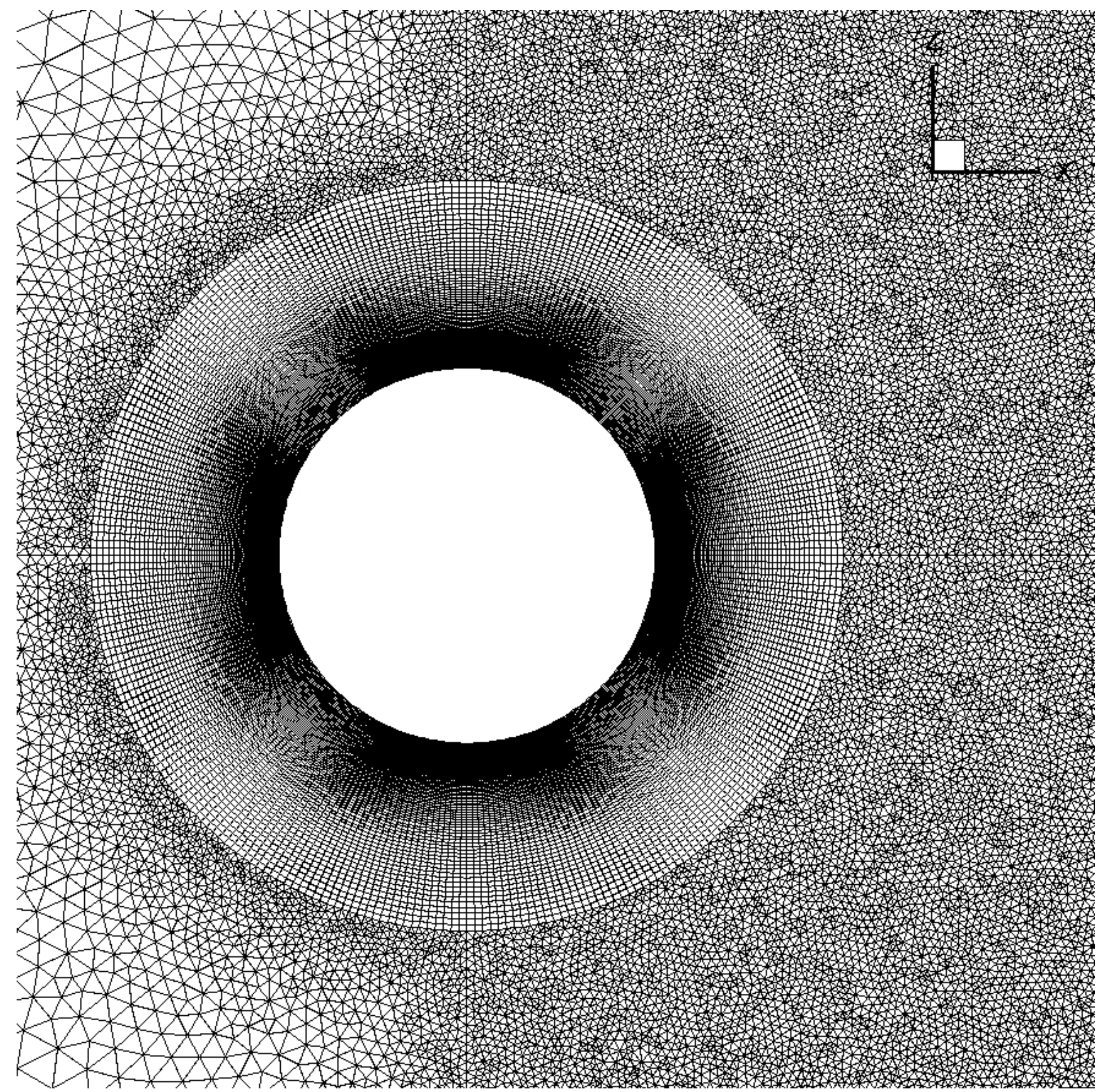
Mixed Grid

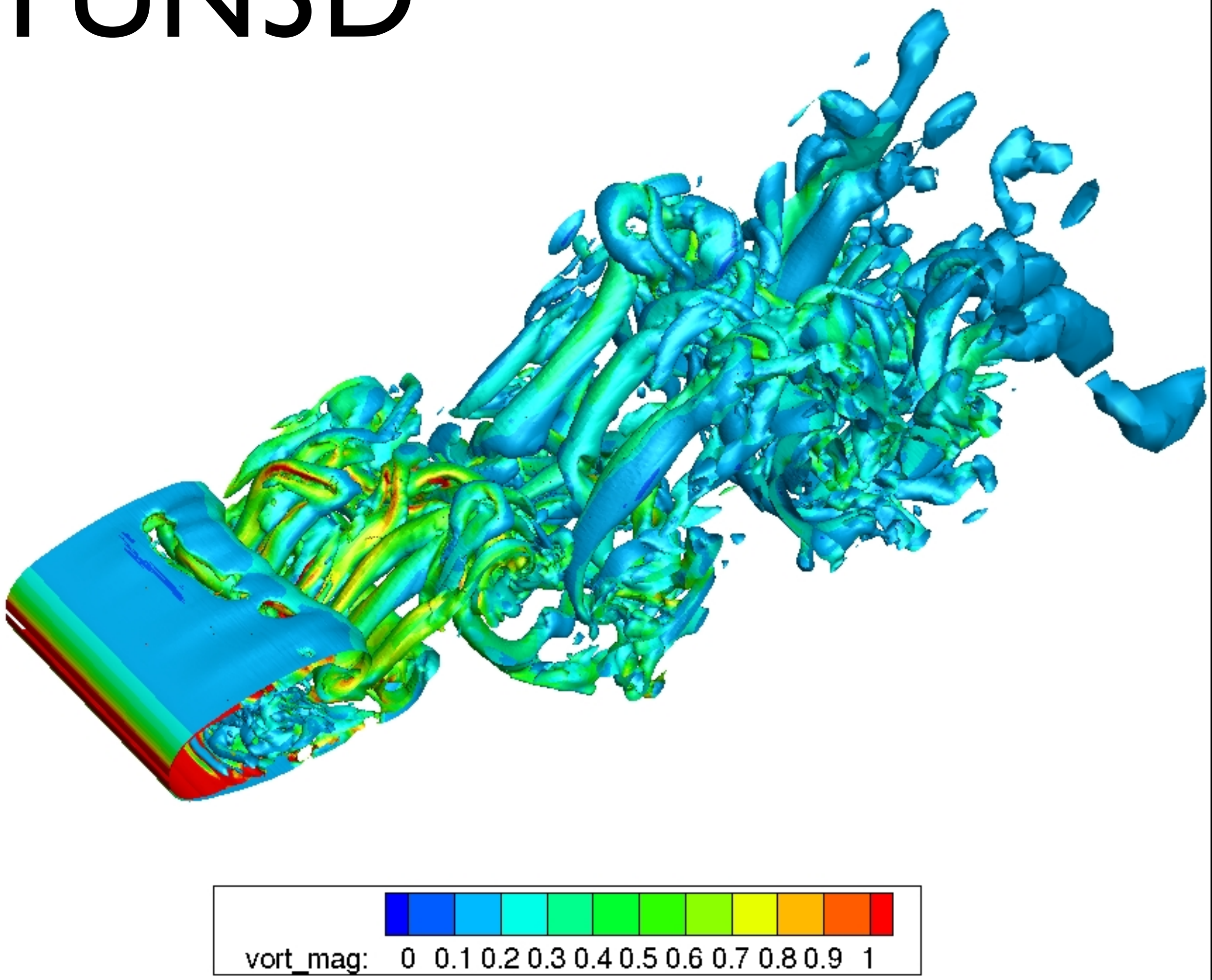




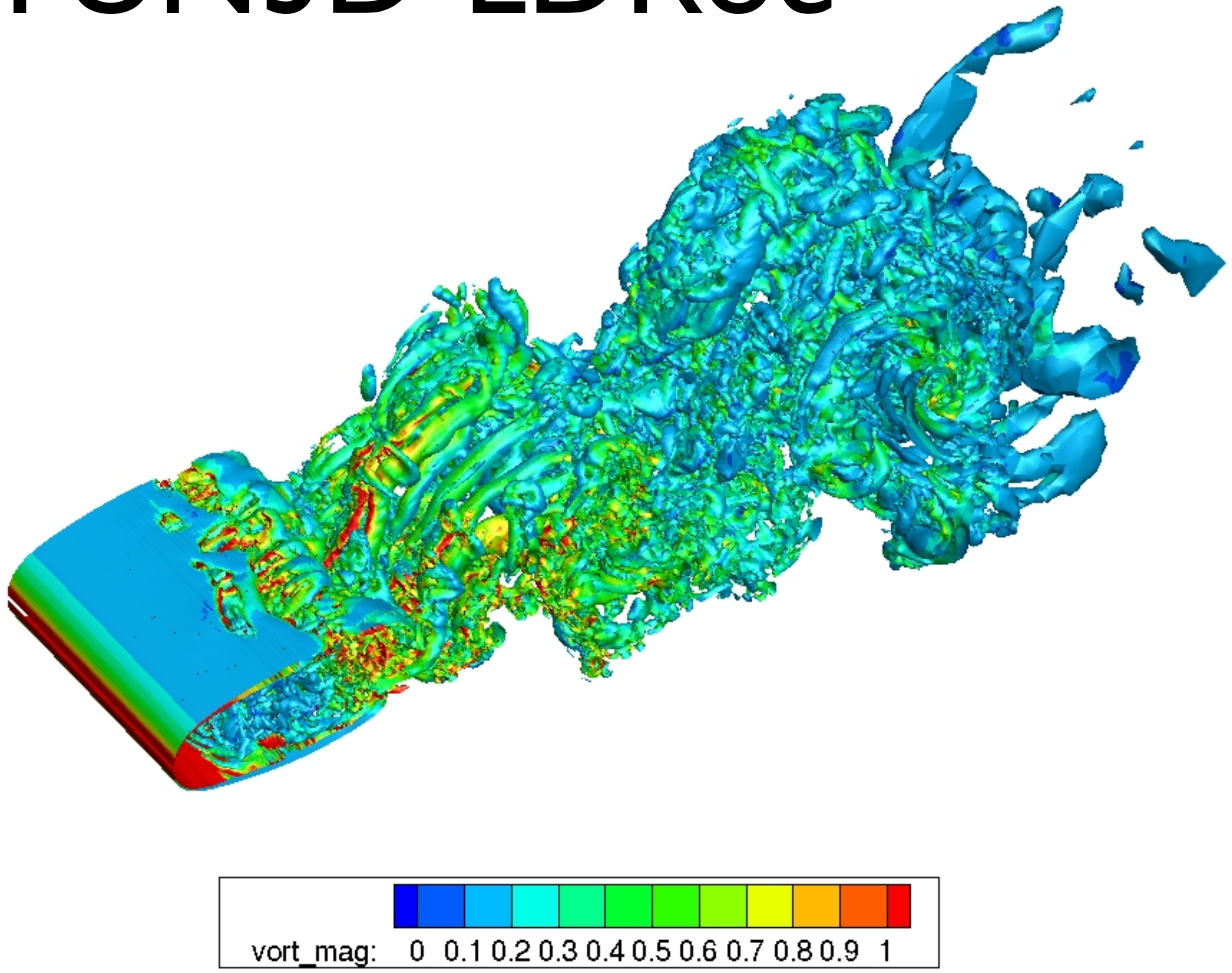
Mixed Grid

Mach=0.1, Re=3900, dt=0.05, 20,000 time steps

FUN3D



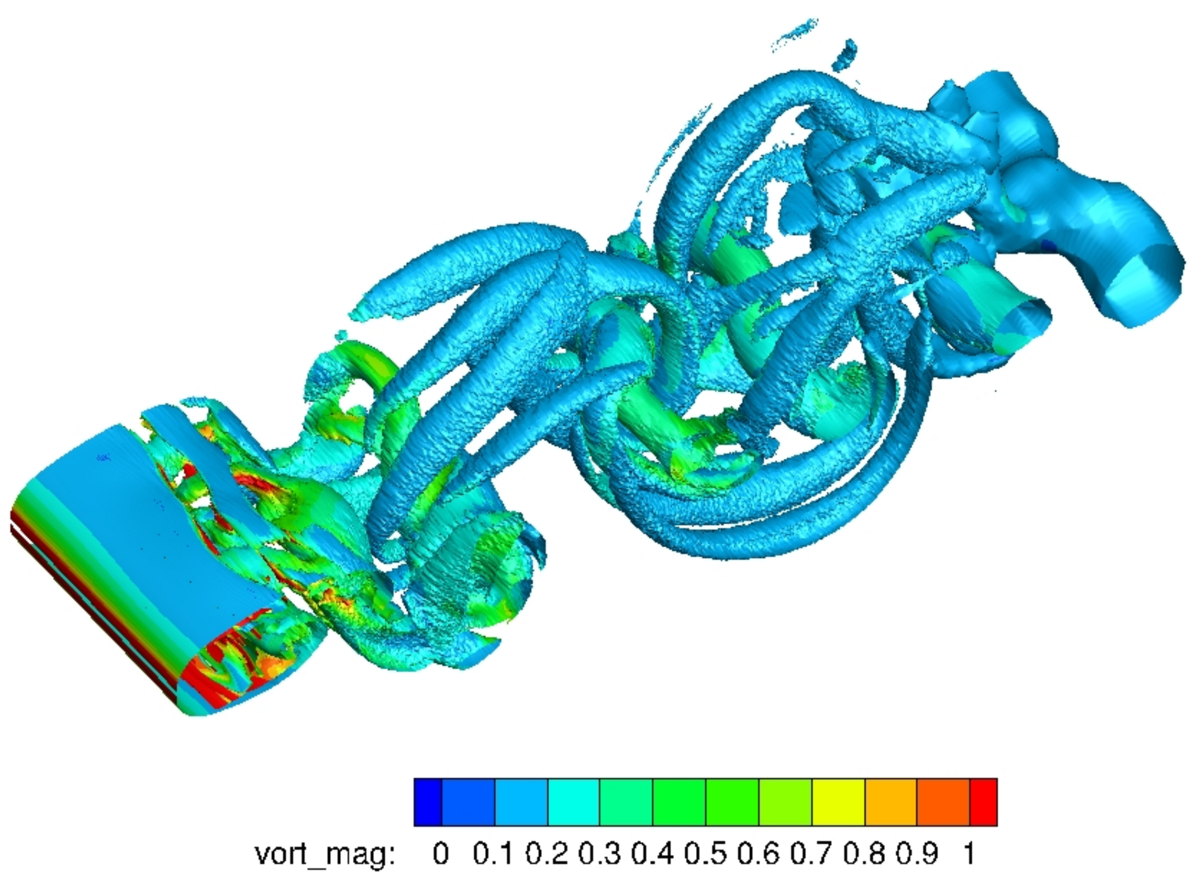
FUN3D-LDRoe



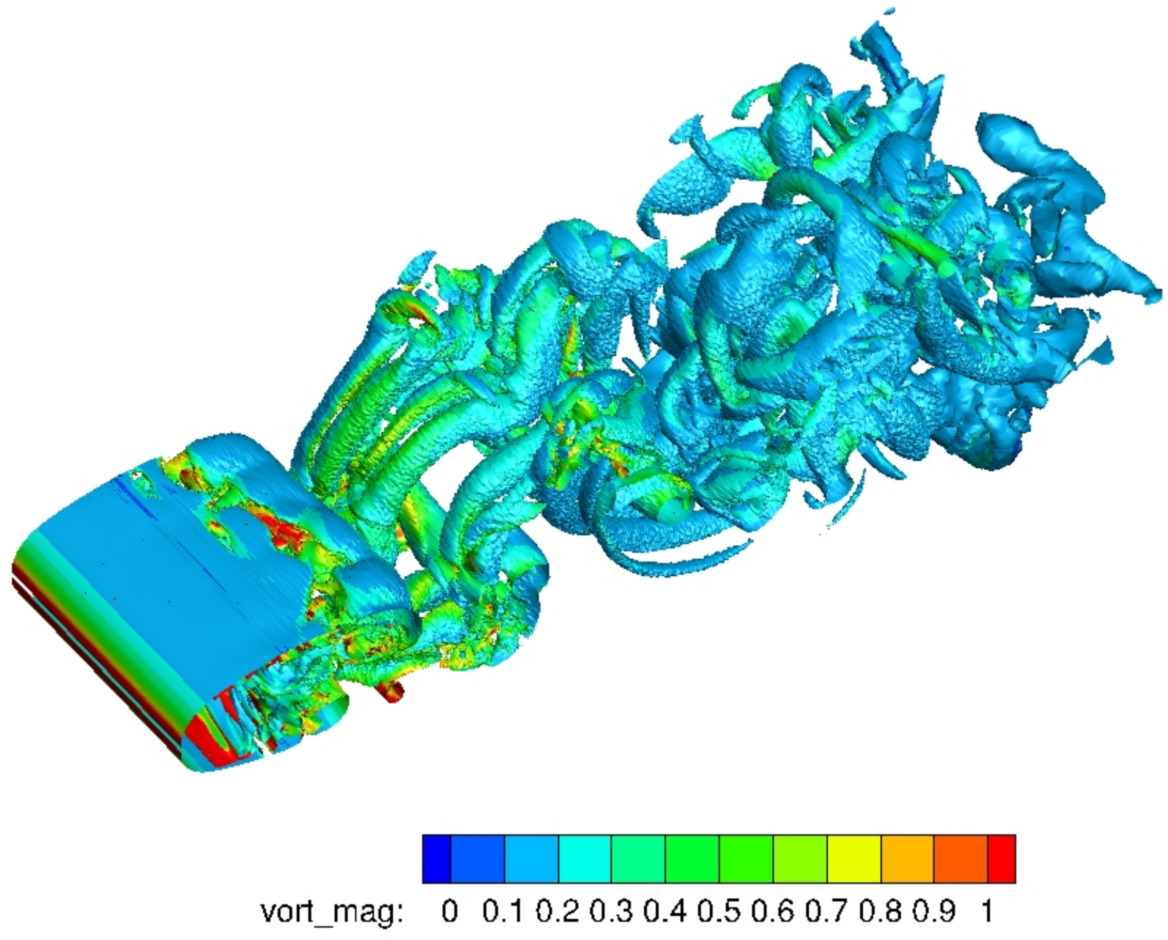
Tetrahedral grid

Mach=0.1, Re=3900, dt=0.05, 20,000 time steps

FUN3D



FUN3D-i3rd-LDRoe



Rotorcraft simulation

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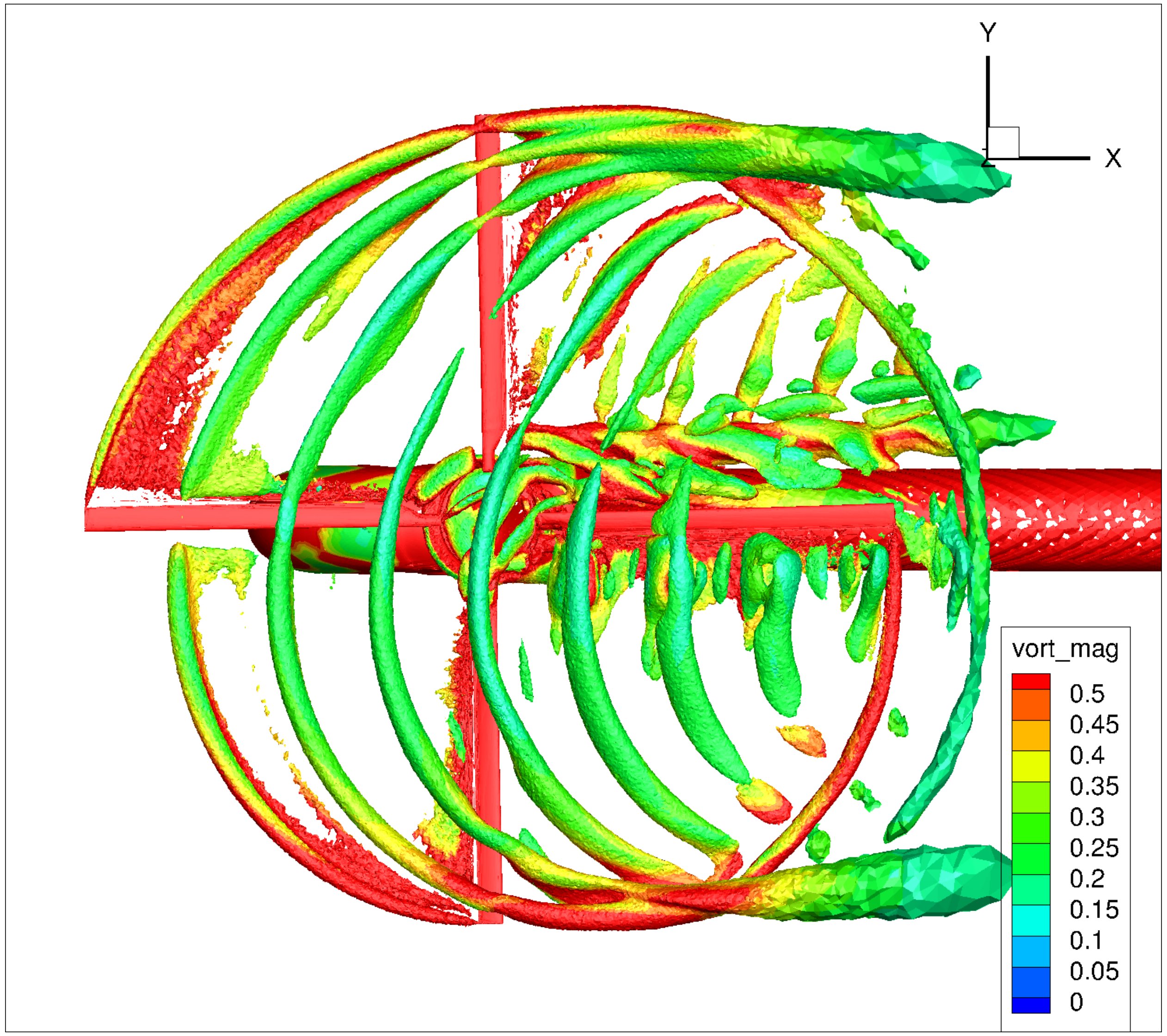
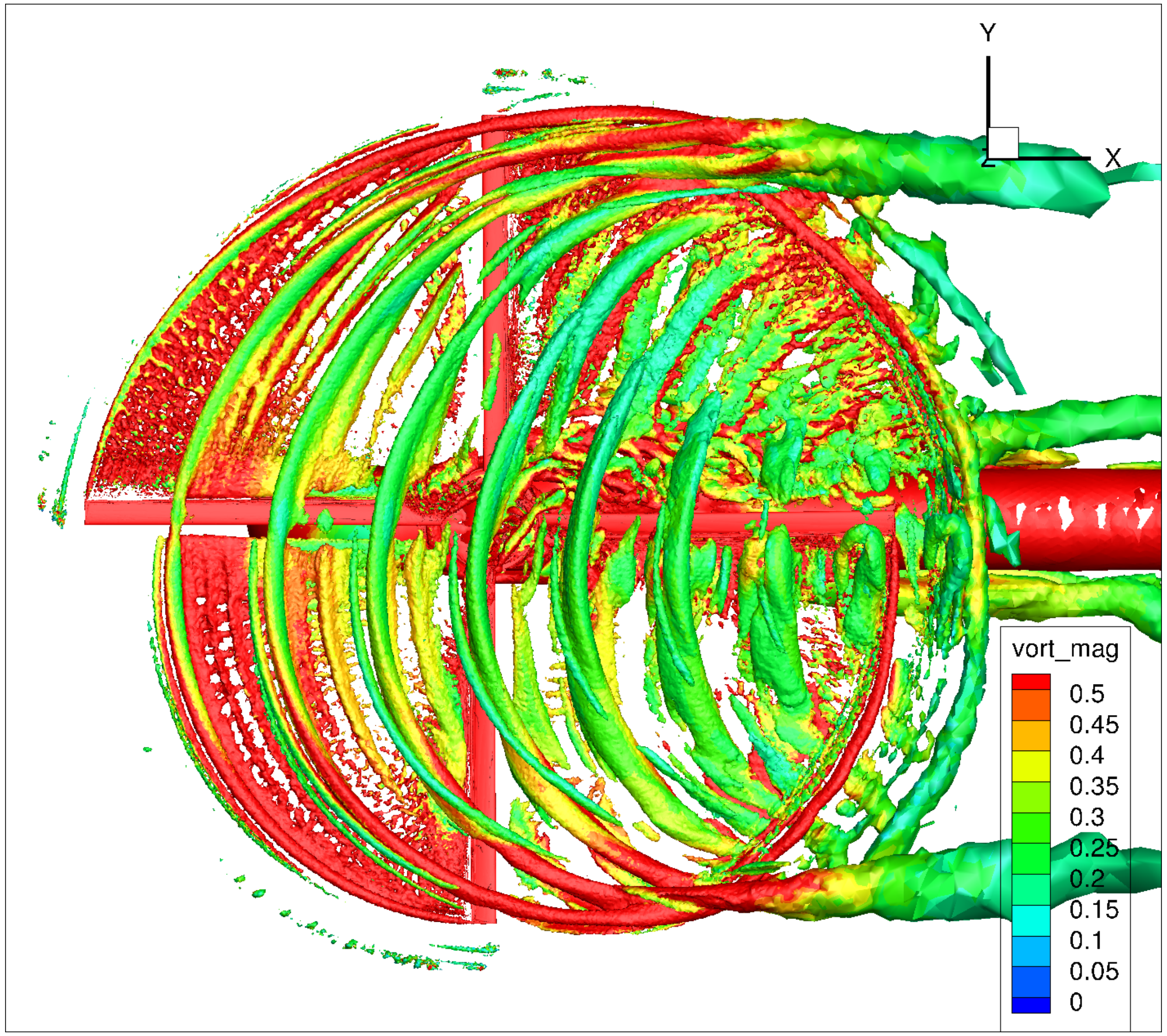
FUN3D



FUN3D-i3rd-LDRoe



Obtained by Beth Lee-Rausch (LaRC)

Conclusions

Better have the same order in space and time.

3rd-order time integration for 3rd-order spatial scheme,
or we may be wasting our resources.

Economical low-dissipation flux

Very cheap and useful if 4th-order scheme is too expensive for you.

3rd-order low-dissipation in NASA's FUN3D

Being used in practical simulations: Rotorcraft, Landing gear, LES, etc.

"Model-invariant hybrid RANS-LES computations on unstructured meshes"
S. Ravindran and S. Woodruff, in this conference.

Future Work

- JFNK solver for 3rd-order scheme
- Implement ESDIRK3 not to waste resources
- Implicit quadratic gradient method
- 3rd-order for turbulence models
- Adaptive viscous simulation with FUN3D-HNS

  Accurate gradients are very useful on adaptive grids and time derivatives
- Enable the use of zero-volume elements

Superior accuracy, efficiency, and immediate applicability warrant further developments.

Shop smart in CFD!

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You can have it now!

Visit the nearest dealership to check out our 3rd-order low-dissipation edge-based scheme!