Hyperbolic Method for Dispersive PDEs:
Same High-Order of Accuracy for Solution, Gradient, and Hessian

Alireza Mazaheri[1], Mario Ricchiuto[2] and Hiroaki Nishikawa[3]

[1] Aerothermodynamics Branch, NASA Langley Research Center
Alireza.Mazaheri@NASA.gov

[2] INRIA Bordeaux Sud-Ouest and Institut de Mathématiques de Bordeaux, France
Mario.Ricchiuto@inria.fr

[3] National Institute of Aerospace
Hiro@nianet.org

71th NIA CFD Seminar
February 24, 2016

Background

Consider the following PDE

$$\underbrace{\partial_t u + \partial_x F(u)}_{Hyperbolic\ model} = \mathcal{D}$$

Background

Consider the following PDE

$$\underbrace{\partial_t u + \partial_x F(u)}_{Hyperbolic\ model} = \mathcal{D}$$

1) Classical viscous regularization $\rightarrow \mathcal{D} = \nu \partial_{xx} u$

Background

Consider the following PDE

$$\underbrace{\partial_t u + \partial_x F(u)}_{Hyperbolic\ model} = \mathcal{D}$$

1) Classical viscous regularization $\rightarrow \mathcal{D} = \nu \partial_{xx} u$
2) Dispersive with time derivative $\rightarrow \mathcal{D} = \epsilon \partial_{xxt} u$

Background

Consider the following PDE

$$\underbrace{\partial_t u + \partial_x F(u)}_{Hyperbolic\ model} = \mathcal{D}$$

1) Classical viscous regularization $\rightarrow \mathcal{D} = \nu \partial_{xx} u$
2) Dispersive with time derivative $\rightarrow \mathcal{D} = \epsilon \partial_{xxt} u$

The PDE can then be decomposed to $\Rightarrow \begin{cases} \partial_t w = -\partial_x f \\ -\partial_{xx} u + u = w \end{cases}$

Background

Consider the following PDE

$$\underbrace{\partial_t u + \partial_x F(u)}_{Hyperbolic\ model} = \mathcal{D}$$

1) Classical viscous regularization $\rightarrow \mathcal{D} = \nu \partial_{xx} u$
2) Dispersive with time derivative $\rightarrow \mathcal{D} = \epsilon \partial_{xxt} u$

The PDE can then be decomposed to $\Rightarrow \begin{cases} \partial_t w = -\partial_x f \\ -\partial_{xx} u + u = w \end{cases}$

- We have already proposed very accurate numerical schemes for cases (1) and (2); e.g., [Hiro, JCP 227(2007), 315–352], [Alireza & Hiro, NASA/TM-2014-218174], [Alireza & Hiro, CF 102 (2014), 131–147], [Hiro, JCP 281(2015), 518–555], [Alireza and Hiro JCP 300(2015), 455–491]

Background

Consider the following PDE

$$\underbrace{\partial_t u + \partial_x F(u)}_{Hyperbolic\ model} = \mathcal{D}$$

1) Classical viscous regularization $\rightarrow \mathcal{D} = \nu \partial_{xx} u$
2) Dispersive with time derivative $\rightarrow \mathcal{D} = \epsilon \partial_{xxt} u$
3) Fully Dispersive $\rightarrow \mathcal{D} = \epsilon \partial_{xxx} u$

Background

Consider the following PDE

$$\underbrace{\partial_t u + \partial_x F(u)}_{Hyperbolic\ model} = \mathcal{D}$$

1) Classical viscous regularization $\rightarrow \mathcal{D} = \nu \partial_{xx} u$
2) Dispersive with time derivative $\rightarrow \mathcal{D} = \epsilon \partial_{xxt} u$
3) Fully Dispersive $\rightarrow \mathcal{D} = \epsilon \partial_{xxx} u$
   - The original hyperbolic advection-diffusion formulation [Hiro, JCP 227(2007), 315–352] will not work for this PDE!

Background

Consider the following PDE

$$\underbrace{\partial_t u + \partial_x F(u)}_{Hyperbolic\ model} = \mathcal{D}$$

1) Classical viscous regularization $\rightarrow \mathcal{D} = \nu \partial_{xx} u$
2) Dispersive with time derivative $\rightarrow \mathcal{D} = \epsilon \partial_{xxt} u$
3) Fully Dispersive $\rightarrow \mathcal{D} = \epsilon \partial_{xxx} u$
   - The original hyperbolic advection-diffusion formulation [Hiro, JCP 227(2007), 315–352] will not work for this PDE! Why?

Background

Consider the following PDE

$$\underbrace{\partial_t u + \partial_x F(u)}_{Hyperbolic\ model} = \mathcal{D}$$

1) Classical viscous regularization $\rightarrow \mathcal{D} = \nu \partial_{xx} u$
2) Dispersive with time derivative $\rightarrow \mathcal{D} = \epsilon \partial_{xxt} u$
3) Fully Dispersive $\rightarrow \mathcal{D} = \epsilon \partial_{xxx} u$
   - The original hyperbolic advection-diffusion formulation [Hiro, JCP 227(2007), 315–352] will not work for this PDE! Why?
   - Because the resulting system will not be hyperbolic [shown first by Toro and Montecinos, SIAM JSC 36(2014), A2423–A2457]

Consider a Fully Dispersive PDE (the 3rd case)

$$\overbrace{\partial_t u}^{physical\ time} = \overbrace{\epsilon\, \partial_{xxx} u}^{dispersion}$$

Consider a Fully Dispersive PDE (the 3rd case)

$$\overbrace{\partial_t u}^{physical\ time} = \overbrace{\epsilon\, \partial_{xxx} u}^{dispersion}$$

- Following JCP2007 reformulation (+ NASA/TM-2014-218175 for time-accurate)

Consider a Fully Dispersive PDE (the 3rd case)

$$\overbrace{\partial_t u}^{physical\ time} = \overbrace{\epsilon\, \partial_{xxx} u}^{dispersion}$$

- Following JCP2007 reformulation (+ NASA/TM-2014-218175 for time-accurate)

$$0 = -\alpha \frac{u}{\Delta t} + s$$

Consider a Fully Dispersive PDE (the 3rd case)

$$\overbrace{\partial_t u}^{physical\ time} = \overbrace{\epsilon\, \partial_{xxx} u}^{dispersion}$$

- Following JCP2007 reformulation (+ NASA/TM-2014-218175 for time-accurate)

$$0 = \epsilon\, \partial_x q \quad -\alpha \frac{u}{\Delta t} + s$$

Consider a Fully Dispersive PDE (the 3rd case)

$$\overbrace{\partial_t u}^{physical\ time} = \overbrace{\epsilon\, \partial_{xxx} u}^{dispersion}$$

- Following JCP2007 reformulation (+ NASA/TM-2014-218175 for time-accurate)

$$0 = \epsilon\, \partial_x q - \alpha \frac{u}{\Delta t} + s$$

$$0 = \frac{1}{T_\epsilon}(\partial_x u - p) \quad \Rightarrow p = u_x$$

$$0 = \frac{1}{T_\epsilon}(\partial_x p - q) \quad \Rightarrow q = p_x = u_{xx}$$

Consider a Fully Dispersive PDE (the 3rd case)

$$\overbrace{\partial_t u}^{physical\ time} = \overbrace{\epsilon\, \partial_{xxx} u}^{dispersion}$$

- Following JCP2007 reformulation (+ NASA/TM-2014-218175 for time-accurate)

$$\partial_\tau u = \epsilon\, \partial_x q - \alpha \frac{u}{\Delta t} + s$$

$$\partial_\tau p = \frac{1}{T_\epsilon}(\partial_x u - p) \qquad \Rightarrow p = u_x$$

$$\partial_\tau q = \frac{1}{T_\epsilon}(\partial_x p - q) \qquad \Rightarrow q = p_x = u_{xx}$$

→ in a vector form:

$$\underbrace{\frac{\partial \mathbf{U}}{\partial \tau}}_{pseudo\ time} + \underbrace{\frac{\partial \mathbf{F}}{\partial x}}_{?} = \underbrace{\mathbf{Q}}_{physical\ time\ +\ all\ sources}$$

Hyperbolic or Not Hyperbolic

- From the previous slide, $\partial_t u$ = $\epsilon\, \partial_{xxx} u$
  $\rightarrow$ in a vector form:

$$\frac{\partial \mathbf{U}}{\partial \tau} + \underbrace{\frac{\partial \mathbf{F}}{\partial x}}_{\boxed{?}} = \mathbf{Q}$$

Hyperbolic or Not Hyperbolic

- From the previous slide, $\partial_t u$ = $\epsilon\, \partial_{xxx} u$
  $\rightarrow$ in a vector form:

$$\frac{\partial \mathbf{U}}{\partial \tau} + \underbrace{\frac{\partial \mathbf{F}}{\partial x}}_{\boxed{?}} = \mathbf{Q}$$

$$\frac{\partial \mathbf{F}}{\partial x} = \mathbf{A}\frac{\partial \mathbf{U}}{\partial x}, \;\; \mathbf{A} = \begin{bmatrix} 0 & 0 & -\epsilon \\ -\dfrac{1}{T_\epsilon} & 0 & 0 \\ 0 & -\dfrac{1}{T_\epsilon} & 0 \end{bmatrix}, \; \mathbf{U} = \begin{bmatrix} u \\ p \rightarrow u_x \\ q \rightarrow u_{xx} \end{bmatrix}$$

Hyperbolic or Not Hyperbolic

- From the previous slide, $\partial_t u$ = $\epsilon\, \partial_{xxx} u$
  $\rightarrow$ in a vector form:

$$\frac{\partial \mathbf{U}}{\partial \tau} + \underbrace{\frac{\partial \mathbf{F}}{\partial x}}_{\boxed{?}} = \mathbf{Q}$$

$$\frac{\partial \mathbf{F}}{\partial x} = \mathbf{A}\frac{\partial \mathbf{U}}{\partial x}, \;\; \mathbf{A} = \begin{bmatrix} 0 & 0 & -\epsilon \\ -\dfrac{1}{T_\epsilon} & 0 & 0 \\ 0 & -\dfrac{1}{T_\epsilon} & 0 \end{bmatrix}, \; \mathbf{U} = \begin{bmatrix} u \\ p \rightarrow u_x \\ q \rightarrow u_{xx} \end{bmatrix}$$

- Eigenvalues:

$$\lambda_1 = -\epsilon^{1/3} T_\epsilon^{-2/3}, \quad \lambda_{2,3} = \frac{1}{2}\lambda_1 (1 \pm \sqrt{3} I)$$

Hyperbolic or Not Hyperbolic

- From the previous slide, $\partial_t u$ = $\epsilon\, \partial_{xxx} u$
  $\rightarrow$ in a vector form:

$$\frac{\partial \mathbf{U}}{\partial \tau} + \underbrace{\frac{\partial \mathbf{F}}{\partial x}}_{\boxed{?}} = \mathbf{Q}$$

$$\frac{\partial \mathbf{F}}{\partial x} = \mathbf{A}\frac{\partial \mathbf{U}}{\partial x}, \quad \mathbf{A} = \begin{bmatrix} 0 & 0 & -\epsilon \\ -\frac{1}{T_\epsilon} & 0 & 0 \\ 0 & -\frac{1}{T_\epsilon} & 0 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} u \\ p \rightarrow u_x \\ q \rightarrow u_{xx} \end{bmatrix}$$

- Eigenvalues:

$$\lambda_1 = -\epsilon^{1/3} T_\epsilon^{-2/3}, \quad \lambda_{2,3} = \frac{1}{2}\lambda_1(1 \pm \sqrt{3}I) \Rightarrow \boxed{\text{Not Hyperbolic!}}$$

Consider (again) a Fully Dispersive PDE (the 3rd case)

$$\overbrace{\partial_t u}^{physical\ time} = \overbrace{\epsilon\, \partial_{xxx} u}^{dispersion}$$

Consider (again) a Fully Dispersive PDE (the 3rd case)

$$\overbrace{\partial_t u}^{physical\ time} = \overbrace{\epsilon\, \partial_{xxx} u}^{dispersion}$$

- Introducing a New Reformulation Strategy

Consider (again) a Fully Dispersive PDE (the 3rd case)

$$\overbrace{\partial_t u}^{physical\ time} = \overbrace{\epsilon\, \partial_{xxx} u}^{dispersion}$$

- Introducing a New Reformulation Strategy

$$0 = -\alpha \frac{u}{\Delta t} + s$$

Consider (again) a Fully Dispersive PDE (the 3rd case)

$$\overbrace{\partial_t u}^{physical\ time} = \overbrace{\epsilon\, \partial_{xxx} u}^{dispersion}$$

- Introducing a New Reformulation Strategy

$$0 = \epsilon\, \partial_x q \quad -\alpha \frac{u}{\Delta t} + s$$

Consider (again) a Fully Dispersive PDE (the 3rd case)

$$\overbrace{\partial_t u}^{physical\ time} = \overbrace{\epsilon\, \partial_{xxx} u}^{dispersion}$$

- Introducing a New Reformulation Strategy

$$0 = \epsilon\, \partial_x q \quad -\alpha \frac{u}{\Delta t} + s$$

$$0 = \frac{1}{T_\epsilon}(\partial_x u - p \qquad)$$

$$0 = \frac{1}{T_\epsilon}(\partial_x p - q \qquad)$$

Consider (again) a Fully Dispersive PDE (the 3rd case)

$$\overbrace{\partial_t u}^{physical\ time} = \overbrace{\epsilon\, \partial_{xxx} u}^{dispersion}$$

- Introducing a New Reformulation Strategy

$$\begin{aligned}
\partial_\tau u &= \epsilon\, \partial_x q - \alpha \frac{u}{\Delta t} + s \\
\partial_\tau p &= \frac{1}{T_\epsilon}(\partial_x u - p) \\
\partial_\tau q &= \frac{1}{T_\epsilon}(\partial_x p - q)
\end{aligned}$$

Consider (again) a Fully Dispersive PDE (the 3rd case)

$$\overbrace{\partial_t u}^{physical\ time} = \overbrace{\epsilon\, \partial_{xxx} u}^{dispersion}$$

- Introducing a New Reformulation Strategy

$$\begin{aligned}
\partial_\tau u &= \epsilon\, \partial_x q \quad -\alpha \frac{u}{\Delta t} + s \\
\partial_\tau p &= \frac{1}{T_\epsilon}(\partial_x u - p \quad -\gamma u) \qquad \Rightarrow p = u_x - \gamma u \\
\partial_\tau q &= \frac{1}{T_\epsilon}(\partial_x p - q \qquad\qquad)
\end{aligned}$$

Consider (again) a Fully Dispersive PDE (the 3rd case)

$$\overbrace{\partial_t u}^{physical\ time} = \overbrace{\epsilon\, \partial_{xxx} u}^{dispersion}$$

- Introducing a New Reformulation Strategy

$$\partial_\tau u = \epsilon\, \partial_x q \; - \alpha \frac{u}{\Delta t} + s$$

$$\partial_\tau p = \frac{1}{T_\epsilon}(\partial_x u - p - \gamma u) \qquad \Rightarrow p = u_x - \gamma u$$

$$\partial_\tau q = \frac{1}{T_\epsilon}(\partial_x p - q + \gamma u_x) \qquad \Rightarrow q = p_x + \gamma u_x = u_{xx}$$

Consider (again) a Fully Dispersive PDE (the 3rd case)

$$\overbrace{\partial_t u}^{physical\ time} = \overbrace{\epsilon\, \partial_{xxx} u}^{dispersion}$$

- Introducing a New Reformulation Strategy

$$\partial_\tau u = \epsilon\, \partial_x q - \alpha \frac{u}{\Delta t} + s$$

$$\partial_\tau p = \frac{1}{T_\epsilon}(\partial_x u - p - \gamma u) \qquad \Rightarrow p = u_x - \gamma u$$

$$\partial_\tau q = \frac{1}{T_\epsilon}(\partial_x p - q + \gamma u_x) \qquad \Rightarrow q = p_x + \gamma u_x = u_{xx}$$

$\rightarrow$ in a vector form:

$$\underbrace{\frac{\partial \mathbf{U}}{\partial \tau}}_{pseudo\ time} + \underbrace{\frac{\partial \mathbf{F}}{\partial x}}_{?} = \underbrace{\mathbf{Q}}_{\substack{physical\ time \\ +\ all\ sources}}$$

New Formulation: Is it Hyperbolic? How?

- $\gamma \neq 0$ is critical

We found...

$$\gamma = \beta + \frac{1}{\epsilon\beta^2 T_\epsilon}, \quad \beta = \frac{\overbrace{\kappa}^{arbitrary\ const.\neq 1}}{L_\epsilon}$$

- The New system has the following Eigenvalues:

$$\lambda_1 = \frac{1}{\beta T_\epsilon}, \quad \lambda_{2,3} = \frac{\lambda_1}{2}\left(-1 \pm \sqrt{1 + 4\epsilon\beta^3 T_\epsilon}\right)$$

- and the following independent Eigenvectors:

$$\mathbf{R} = \begin{bmatrix} -\epsilon & -\lambda_2 T_\epsilon & -\lambda_3 T_\epsilon \\ \epsilon\beta & 1 & 1 \\ \lambda_1 & \beta - \lambda_2/(\epsilon\beta) & \beta - \lambda_3/(\epsilon\beta) \end{bmatrix} \Rightarrow \boxed{\text{Is Hyperbolic!}}$$

Dispersion Relaxation Time

$$T_\epsilon = \frac{Dispersion\ Length}{Wave\ Length} = \frac{L_\epsilon}{\lambda_2}$$

- Note $\lambda = \lambda(\beta), \quad \beta = \beta(\kappa)$

Choice of Arbitrary Constant κ

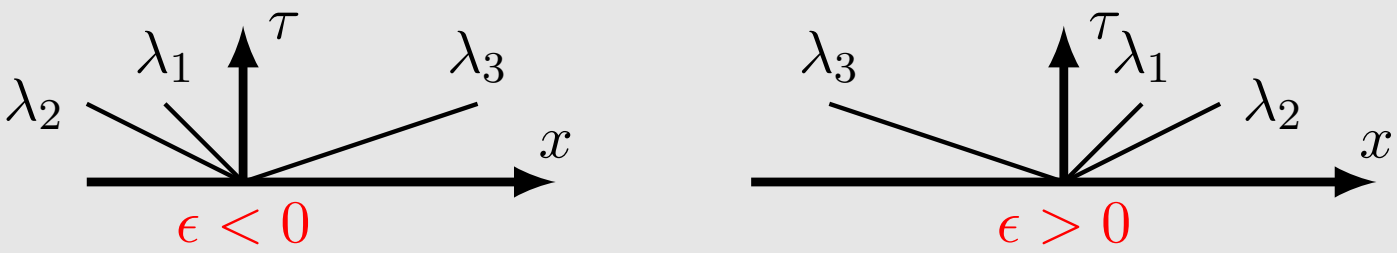
- any value except $\kappa = 1$ is a valid value
- $\kappa = 1 \rightarrow$ singular eigenvector
- we recommend

$$\kappa = \kappa_g \equiv (1 + \sqrt{5})/2$$

which makes everything simpler:

$$T_\epsilon = \frac{L_\epsilon^3}{\epsilon}$$

New Hyperbolic System for Dispersion



$$\lambda_1 = \frac{L_\epsilon}{\kappa_g T_\epsilon}, \quad \lambda_2 = \kappa_g \lambda_1, \quad \lambda_3 = -\kappa_g^2 \lambda_1,$$

The γ parameter can now be fully defined as

$$\gamma = \beta + \frac{1}{\epsilon \beta^2 T_\epsilon} = \boxed{\frac{2}{L_\epsilon}}$$

Dispersion Length Scale

- Fourier Analysis: $\rightarrow \mathbf{U} = \mathbf{U}_0 e^{I\theta h} \quad \rightarrow \quad \frac{\partial \mathbf{U}}{\partial \tau} + \frac{\partial \mathbf{F}}{\partial x} = \mathbf{Q}$
- Eigenvalues of Fourier-Transformed Operator:

$$\lambda_1 = \overbrace{-\frac{I\epsilon}{h^3}\theta^3}^{Fourier\ Symbol\ of\ \epsilon\partial_{xxx}u} + \frac{2I\epsilon L_\epsilon^2}{h^5}\theta^5 + \frac{2\epsilon L_\epsilon^3}{h^6}\theta^6 + O(\theta^7),$$

$$\lambda_{2/3} = \underbrace{-\frac{\epsilon}{L_\epsilon^3}}_{Damping\ Mode} \pm \underbrace{\frac{I\sqrt{2}\,\epsilon}{L_\epsilon^2 h}\theta}_{Propagation\ Mode} + O(\theta^2).$$

- For the slowest frequency error mode ($\theta = \pi h$):

$$\frac{\epsilon}{h^3}\theta^3 = \frac{\sqrt{2}\,\epsilon}{L_\epsilon^2 h}\theta \Rightarrow L_\epsilon = \pm\frac{2^{\frac{1}{4}}}{\pi}$$

Hyperbolic Advection-Diffusion-Dispersion

Consider a general Advective-Diffusive-Dispersive PDE

$$\overbrace{\partial_t u}^{physical\ time} + \overbrace{\partial_x(f)}^{advection} = \overbrace{\partial_x(\nu \partial_x u)}^{diffusion} + \overbrace{\epsilon\, \partial_{xxx} u}^{dispersion}$$

Hyperbolic Advection-Diffusion-Dispersion

Consider a general Advective-Diffusive-Dispersive PDE

$$\overbrace{\partial_t u}^{\textit{physical time}} + \overbrace{\partial_x(f)}^{\textit{advection}} = \overbrace{\partial_x(\nu \partial_x u)}^{\textit{diffusion}} + \overbrace{\epsilon\, \partial_{xxx} u}^{\textit{dispersion}}$$

$$\partial_\tau u = \qquad\qquad\qquad\qquad +\epsilon\, \partial_x q \quad -\alpha \frac{u}{\Delta t} + s$$

$$\partial_\tau p = \left(\frac{1}{T_\epsilon} \qquad\qquad\qquad\qquad\right)(\partial_x u - p - \gamma u) \quad \Rightarrow p = \partial_x u - \gamma u$$

$$\partial_\tau q = \frac{1}{T_\epsilon}(\partial_x p - q + \gamma u_x) \qquad \Rightarrow q = p_x + \gamma u_x = u_{xx}$$

Hyperbolic Advection-Diffusion-Dispersion

Consider a general Advective-Diffusive-Dispersive PDE

$$\overbrace{\partial_t u}^{\textit{physical time}} + \overbrace{\partial_x(f)}^{\textit{advection}} = \overbrace{\partial_x(\nu \partial_x u)}^{\textit{diffusion}} + \overbrace{\epsilon\, \partial_{xxx} u}^{\textit{dispersion}}$$

$$\partial_\tau u = -\partial_x(f) \qquad +\epsilon\, \partial_x q \qquad -\alpha \frac{u}{\Delta t} + s$$

$$\partial_\tau p = \left(\frac{1}{T_\epsilon} \qquad\qquad\right)(\partial_x u - p - \gamma u) \quad \Rightarrow p = \partial_x u - \gamma u$$

$$\partial_\tau q = \frac{1}{T_\epsilon}(\partial_x p - q + \gamma u_x) \qquad \Rightarrow q = p_x + \gamma u_x = u_{xx}$$

Hyperbolic Advection-Diffusion-Dispersion

Consider a general Advective-Diffusive-Dispersive PDE

$$\overbrace{\partial_t u}^{\textit{physical time}} + \overbrace{\partial_x(f)}^{\textit{advection}} = \overbrace{\partial_x(\nu \partial_x u)}^{\textit{diffusion}} + \overbrace{\epsilon\, \partial_{xxx} u}^{\textit{dispersion}}$$

$$\partial_\tau u = -\partial_x(f) + \partial_x(\nu p + \gamma \nu u) + \epsilon\, \partial_x q - \alpha \frac{u}{\Delta t} + s$$

$$\partial_\tau p = \left(\frac{1}{T_\epsilon} \qquad \right)(\partial_x u - p - \gamma u) \quad \Rightarrow p = \partial_x u - \gamma u$$

$$\partial_\tau q = \frac{1}{T_\epsilon}(\partial_x p - q + \gamma u_x) \qquad \Rightarrow q = p_x + \gamma u_x = u_{xx}$$

Hyperbolic Advection-Diffusion-Dispersion

Consider a general Advective-Diffusive-Dispersive PDE

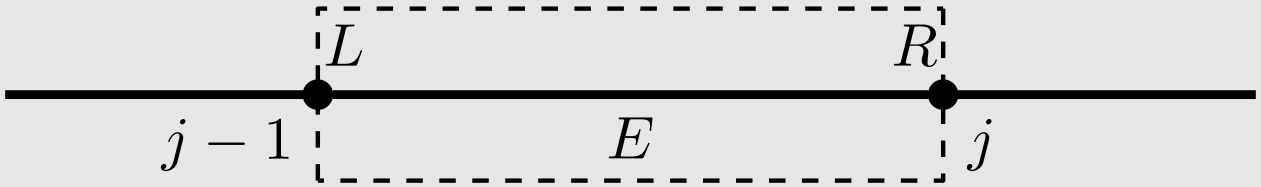
$$\overbrace{\partial_t u}^{\textit{physical time}} + \overbrace{\partial_x(f)}^{\textit{advection}} = \overbrace{\partial_x(\nu \partial_x u)}^{\textit{diffusion}} + \overbrace{\epsilon\, \partial_{xxx} u}^{\textit{dispersion}}$$

$$\partial_\tau u = -\partial_x(f) + \partial_x(\nu p + \gamma \nu u) + \epsilon\, \partial_x q - \alpha \frac{u}{\Delta t} + s$$

$$\partial_\tau p = \left(\frac{1}{T_\epsilon} + \frac{1}{T_\nu}\right)(\partial_x u - p - \gamma u) \Rightarrow p = \partial_x u - \gamma u$$

$$\partial_\tau q = \frac{1}{T_\epsilon}(\partial_x p - q + \gamma u_x) \qquad \Rightarrow q = p_x + \gamma u_x = u_{xx}$$

Discretization: Residual-Distribution



- Cell Residual

$$\mathbf{\Phi}^E = \int_j^{j+1} \left(-\mathbf{A}\partial_x \mathbf{U} + \mathbf{Q}\right) dx$$

- Residual Distribution

$$\mathbf{\Phi}^{\mathsf{L}} = \mathbf{B}^- \, \mathbf{\Phi}^{\mathbf{E}}, \quad \mathbf{\Phi}^{\mathsf{R}} = \mathbf{B}^+ \, \mathbf{\Phi}^{\mathbf{E}}$$

Discretization: Residual-Distribution

- SUPG Distribution

$$\mathbf{B}^{\pm} = \underbrace{\frac{1}{2}\mathbf{I}}_{Galerkin} \pm \underbrace{\mathbf{D}^{\text{adv}} \pm \mathbf{D}^{\text{diff}} \pm \mathbf{D}^{\text{disp}}}_{Stabilization}$$

$$\mathbf{B}^{\pm} = \frac{1}{2}\mathbf{R}\begin{bmatrix} 1 \pm \frac{\lambda_1}{|\lambda_1|} & 0 & 0 \\ 0 & 1 \pm \frac{\lambda_2}{|\lambda_2|} & 0 \\ 0 & 0 & 1 \pm \frac{\lambda_3}{|\lambda_3|} \end{bmatrix}\mathbf{L}$$

$$= \frac{1}{2}\mathbf{I} \pm \frac{1}{2}\mathbf{A}\left(\mathbf{R}|\mathbf{\Lambda}|^{-1}\mathbf{L}\right) = \underbrace{\frac{1}{2}\mathbf{I}}_{Galerkin} \pm \underbrace{\frac{1}{2}\mathbf{A}\sum_{l=1}^{3}\frac{1}{|\lambda_l|}\mathbf{\Pi}_l}_{Stabilization},$$

Discretization: Residual-Distribution

- SUPG Stabilization

$$\underbrace{\mathbf{B}^{\pm}}_{Residual\ Distribution} = \underbrace{\frac{1}{2}\mathbf{I}}_{Galerkin} \pm \underbrace{\mathbf{D}^{\text{adv}} \pm \mathbf{D}^{\text{diff}} \pm \mathbf{D}^{\text{disp}}}_{Stabilization}$$

$$= \underbrace{\frac{1}{2}\mathbf{I}}_{Galerkin} \pm \underbrace{\frac{1}{2}\mathbf{A}\sum_{l=1}^{3}\frac{1}{|\lambda_l|}\mathbf{\Pi}_l}_{Stabilization}$$

Discretization: Residual-Distribution

- SUPG Stabilization

$$\underbrace{\mathbf{B}^{\pm}}_{Residual\ Distribution} = \underbrace{\frac{1}{2}\mathbf{I}}_{Galerkin} \pm \underbrace{\mathbf{D}^{\text{adv}} \pm \mathbf{D}^{\text{diff}} \pm \mathbf{D}^{\text{disp}}}_{Stabilization}$$

$$= \underbrace{\frac{1}{2}\mathbf{I}}_{Galerkin} \pm \underbrace{\frac{1}{2}\mathbf{A}\sum_{l=1}^{3}\frac{1}{|\lambda_l|}\mathbf{\Pi}_l}_{Stabilization}$$

$$\mathbf{A}^{\text{adv}} + \mathbf{A}^{\text{diff}} + \mathbf{A}^{\text{disp}}$$

Discretization: Residual-Distribution

- SUPG Stabilization

$$\underbrace{\mathbf{B}^{\pm}}_{Residual\ Distribution} = \underbrace{\frac{1}{2}\mathbf{I}}_{Galerkin} \pm \underbrace{\mathbf{D}^{\text{adv}} \pm \mathbf{D}^{\text{diff}} \pm \mathbf{D}^{\text{disp}}}_{Stabilization}$$

$$= \underbrace{\frac{1}{2}\mathbf{I}}_{Galerkin} \pm \underbrace{\frac{1}{2}\mathbf{A}\sum_{l=1}^{3}\frac{1}{|\lambda_l|}\mathbf{\Pi}_l}_{Stabilization}$$

- $\Rightarrow$ Individual Stabilization terms are:

$$\mathbf{D}^{\text{adv}} = \frac{1}{2}\mathbf{A}^{\text{adv}}/(|a-\gamma\nu| + \tilde{\epsilon}), \quad \tilde{\epsilon} \ll 1$$

$$\mathbf{D}^{\text{diff}} = \frac{1}{2}\mathbf{A}^{\text{diff}}\sum_{l=1}^{2}\frac{1}{|\lambda_l^{\text{diff}}|}\mathbf{\Pi}_l^{\text{diff}}, \quad \mathbf{D}^{\text{disp}} = \frac{1}{2}\mathbf{A}^{\text{disp}}\sum_{l=1}^{3}\frac{1}{|\lambda_l^{\text{disp}}|}\mathbf{\Pi}_l^{\text{disp}}$$

Two-Soliton Korteweg-de Vries (KdV)

Consider a general time-dependent dispersive PDE

$$\partial_t u + \partial_x f = \partial_x(\nu \partial_x u) + \epsilon\, \partial_{xxx} u + \tilde{s}(x,u), \quad f = 3u^2,\ \nu = 0.5,\ \epsilon = 1$$

Method of Manufactured Solution

$$u^e(x,t) = \frac{(\eta_2 - \eta_1)\left(\eta_1\, sech^2\left[\chi(\eta_1)\right] + \eta_2\, csch^2\left[\chi(\eta_2)\right]\right)}{\left(\sqrt{\eta_1}\, tanh\left[\chi(\eta_1)\right] - \sqrt{\eta_2}\, coth\left[\chi(\eta_2)\right]\right)^2},$$

$$\chi(\eta) = \sqrt{\frac{\eta}{2}}\,(x - 2\,\eta\, t - \tilde{a})$$

$$\eta_1 = 0.5, \quad \eta_2 = 1.0, \quad \tilde{a} = 1.0$$

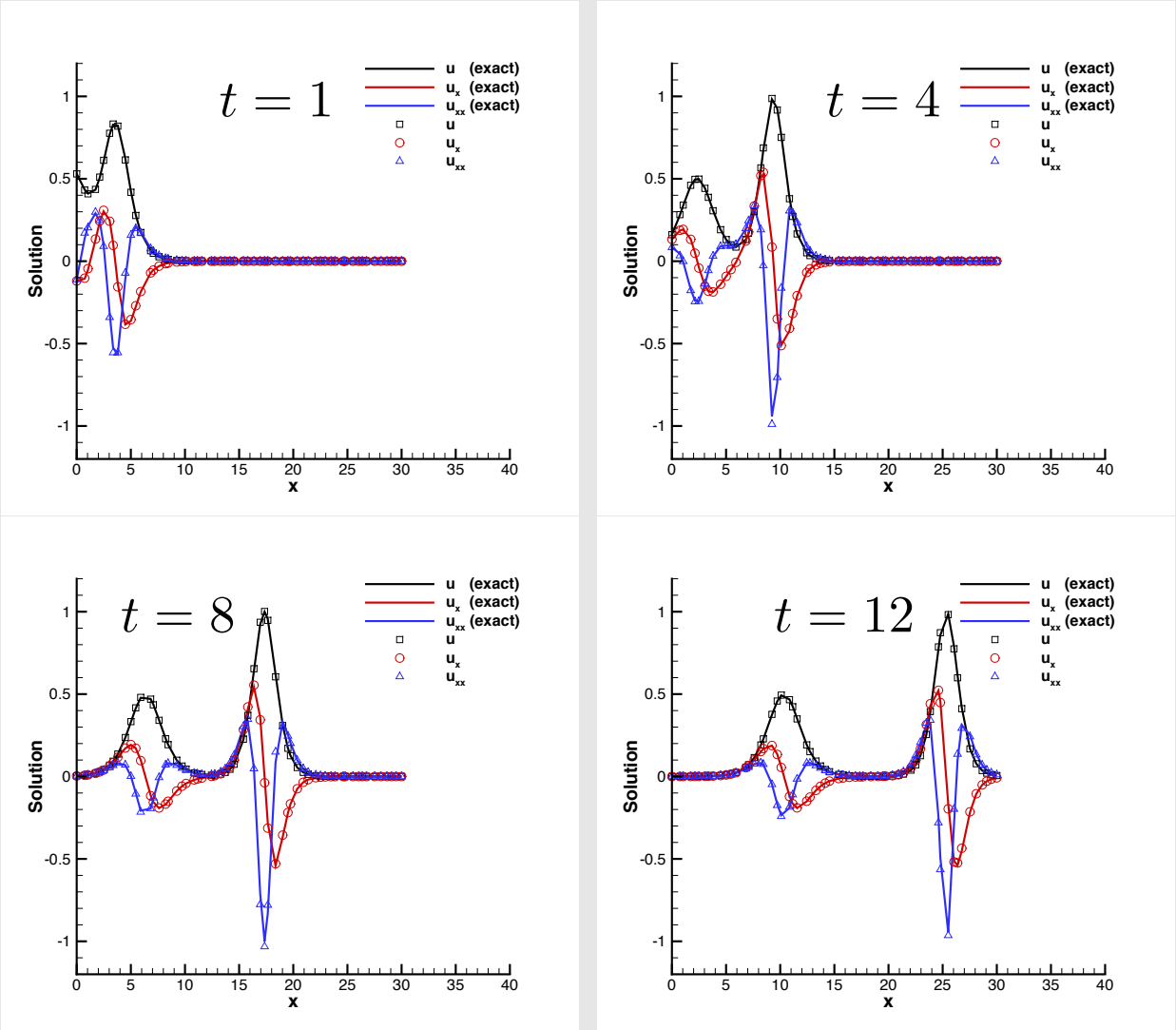
Boundary Condition & Grid

Dirichlet Boundary with Randomly Distributed Grid Points

Two-Soliton Korteweg-de Vries (KdV):

Fourth-Order RD Scheme [Alireza & Hiro, CF 102(2014), 131–147]

on $N = 60 \in x[0, 30]$ with BDF2, $\Delta t = 0.01$

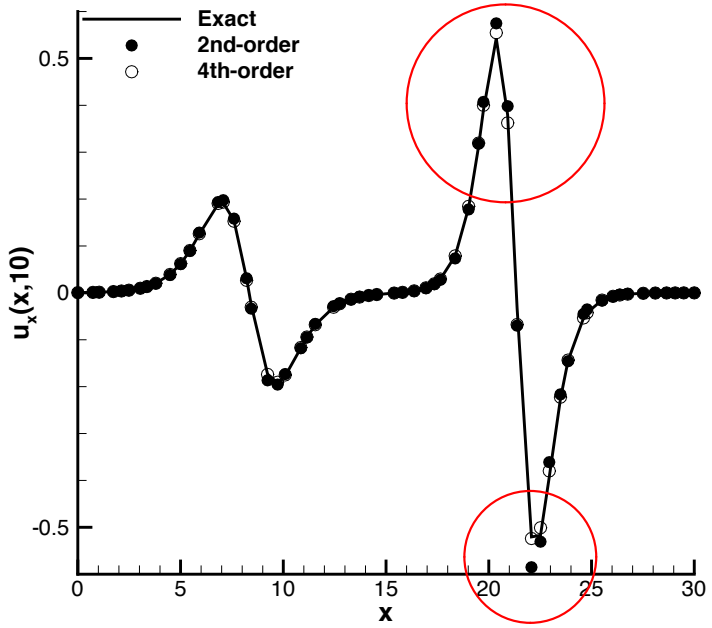


Two-Soliton Korteweg-de Vries (KdV):

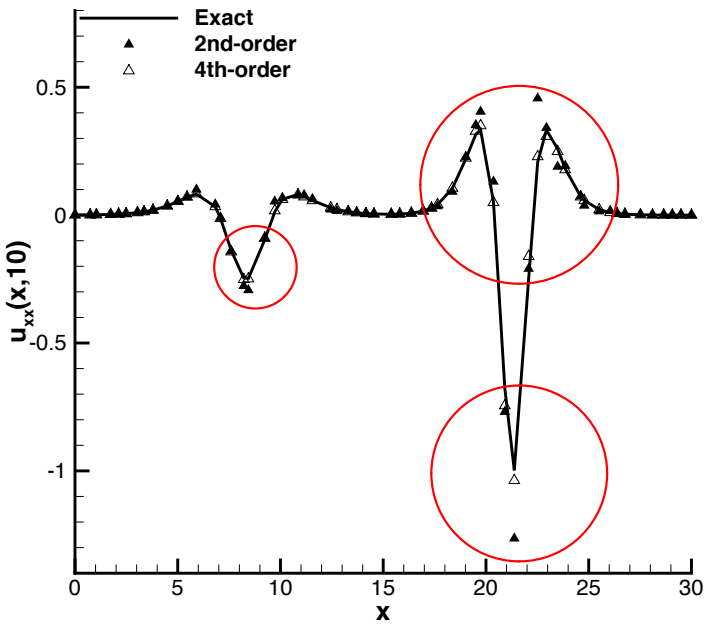
Fourth-Order vs. Second-Order [Alireza & Hiro, CF 102(2014), 131–147]

on $N = 60 \in x[0, 30]$ with BDF2, $\Delta t = 0.001$

Gradient, $u_x(x, \underline{t = 10})$

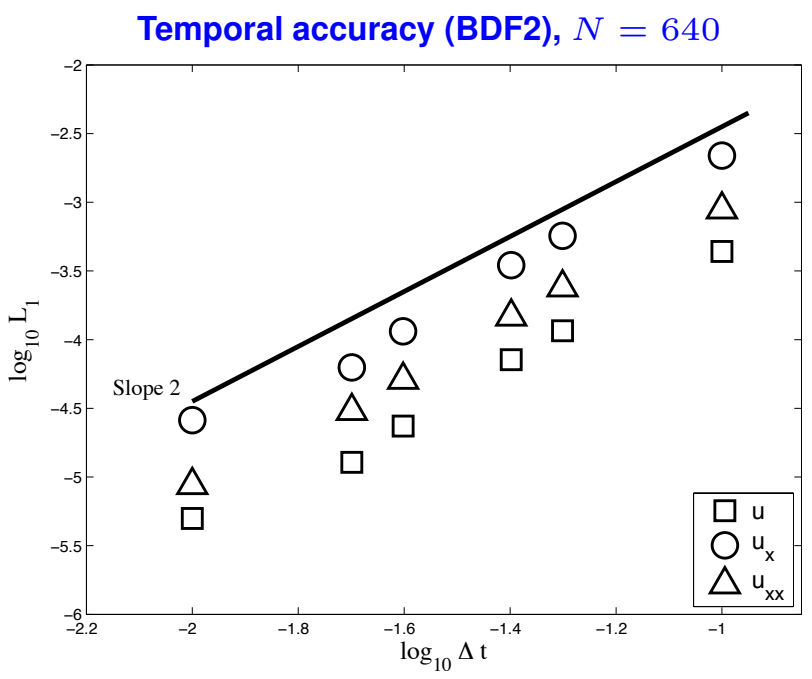


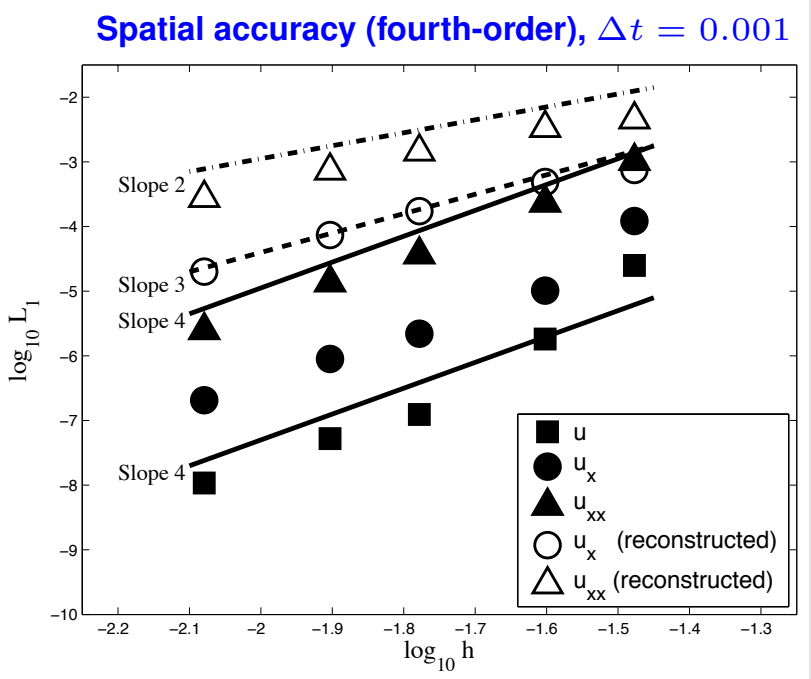
Hessian, $u_{xx}(x, \underline{t = 10})$



4th-order scheme accurately detects the peaks and valleys.

Accuracy Verification on Randomly Distributed Grid Points





- Equal order of accuracy (4th-order) for u, u_x & u_{xx}

Korteweg-de Vries (KdV)

Consider a time-dependent KdV equation

$$\partial_t u + \partial_x f = \epsilon\, \partial_{xxx} u, \quad f = u^2/2, \quad \epsilon = -4.84 \times 10^{-4}$$

Initial Condition

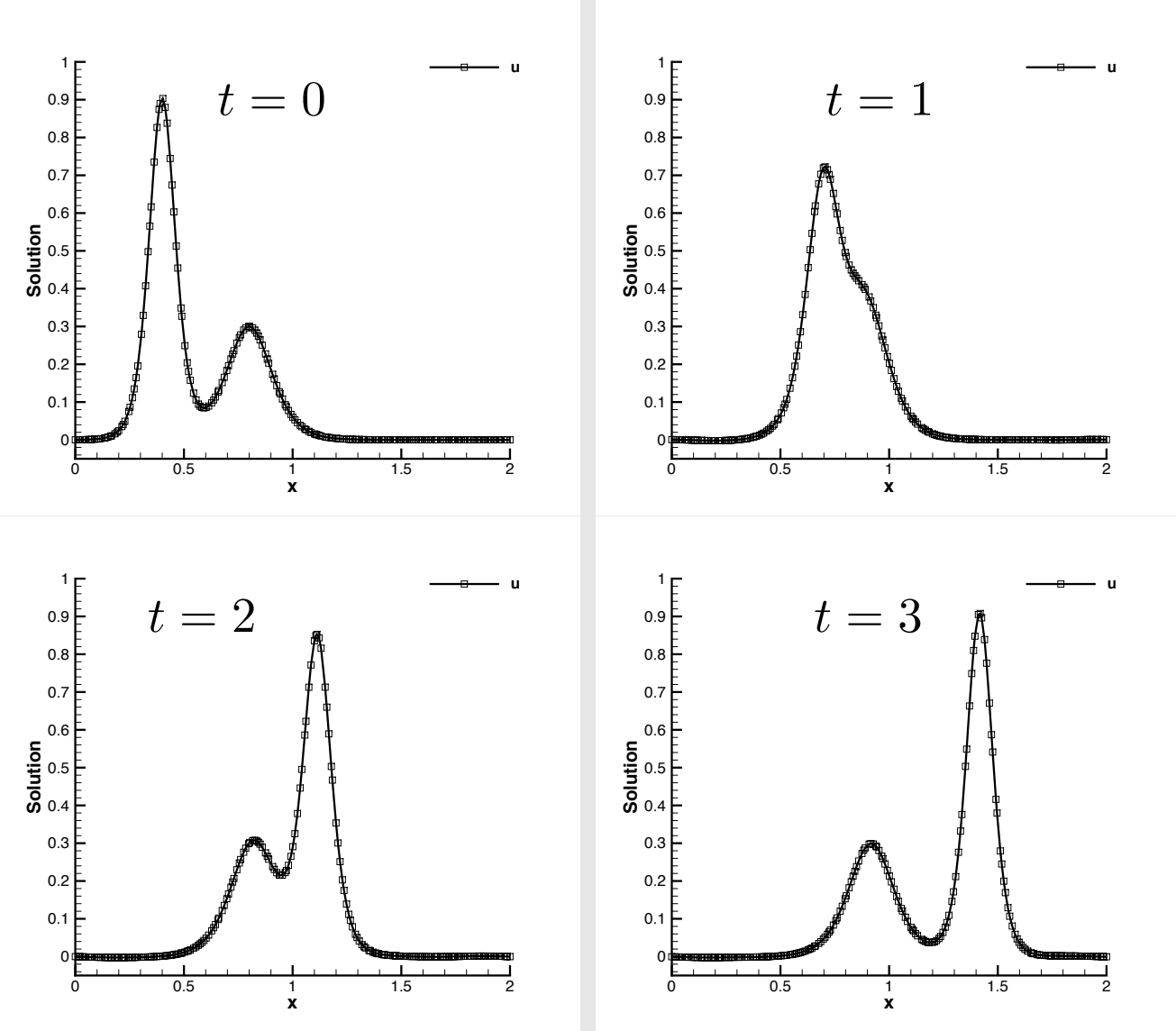
$$u(x,0) = 3\,\eta_1\, sech^2\left(\frac{1}{2}\sqrt{-\eta_1/\epsilon}\,[(x - x_1) - \eta_1 t]\right) + 3\,\eta_2\, sech^2\left(\frac{1}{2}\sqrt{-\eta_2/\epsilon}\,[(x - x_2) - \eta_2 t]\right)$$

$$\eta_1 = 0.3, \quad \eta_2 = 0.1, \quad x_1 = 0.4, \quad x_2 = 0.8$$

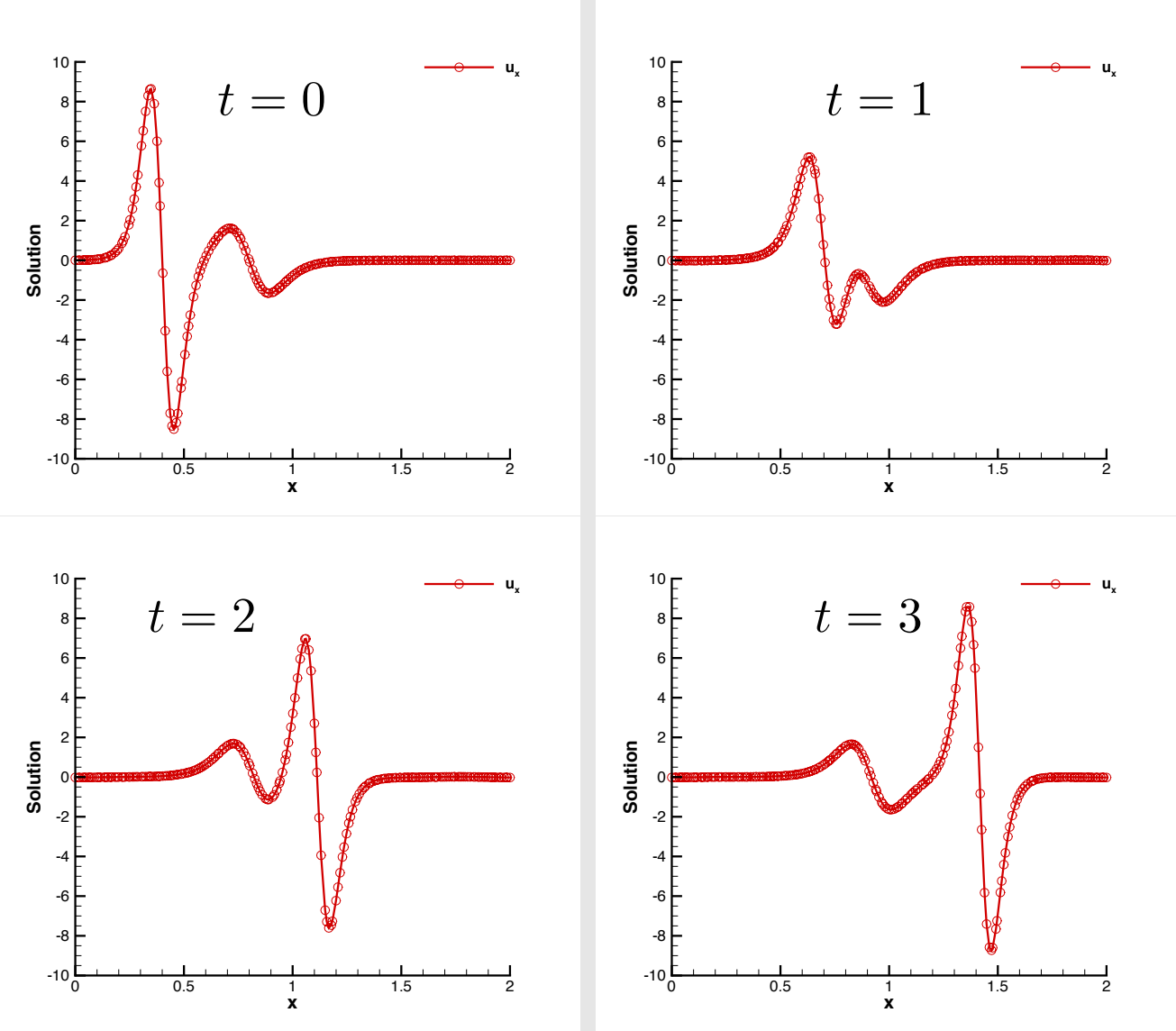
Boundary Condition & Grid

Periodic Boundary with Randomly Distributed Grid Points

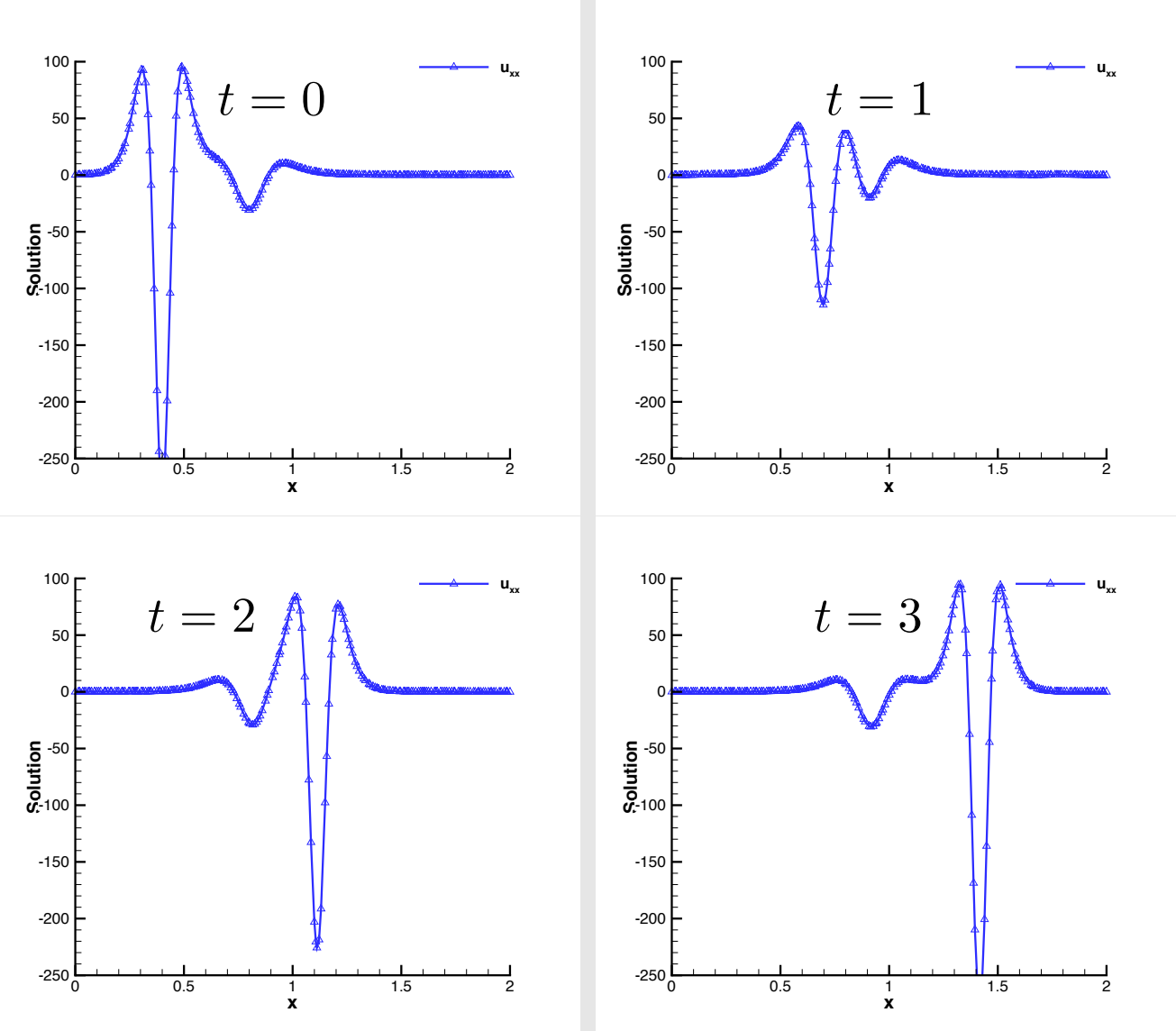
Korteweg-de Vries (KdV): Solution u



Korteweg-de Vries (KdV): Solution Gradient u_x



Korteweg-de Vries (KdV): Hessian u_{xx}



Dispersive Shock (Zero Dispersion Limit)

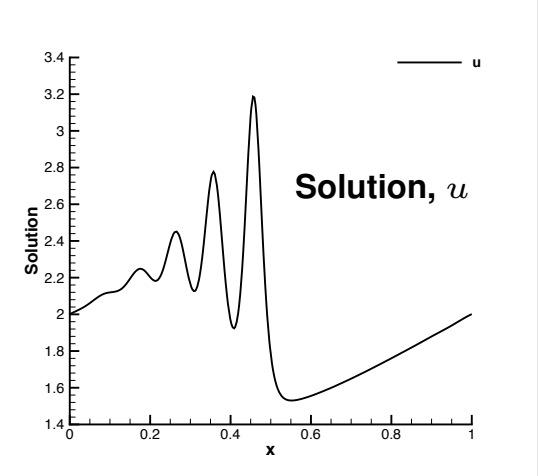
Consider a time-dependent Dispersive Nonlinear Burgers equation

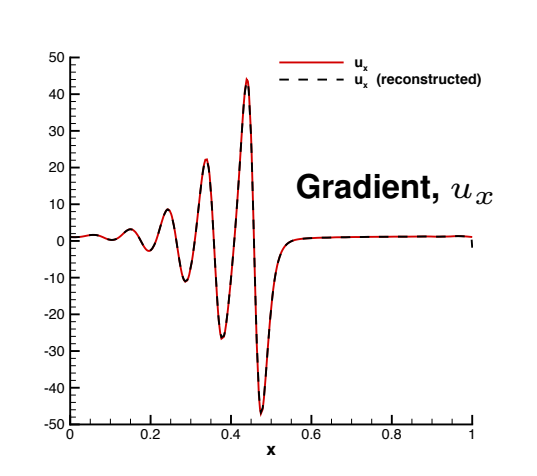
$$\partial_t u + u\partial_x u = \epsilon\, \partial_{xxx} u, \quad \epsilon \to 0^+$$

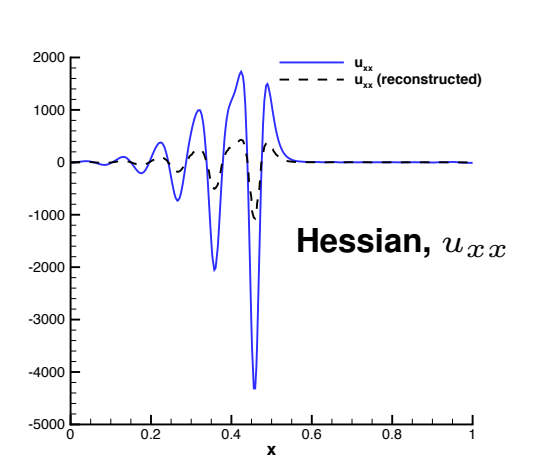
Behavior of this equation as $\epsilon \to 0^+$

- Continuous wavelets in the vicinity of the discontinuity (aka Dispersive Shock Waves)
- The smaller the $\epsilon > 0$, the higher the frequency of the wavelets
- Known as an extremely difficult case

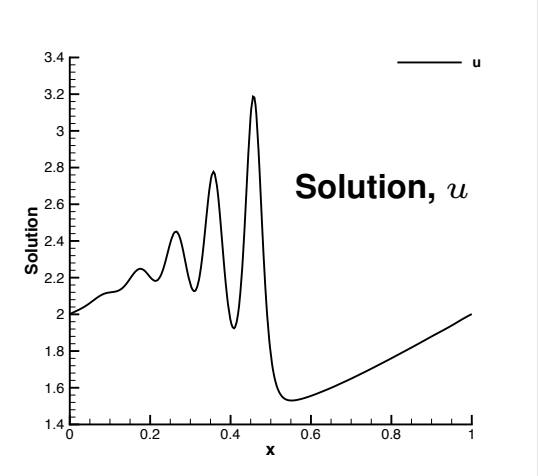
Zero Dispersion Limit: $\epsilon = 10^{-4}$

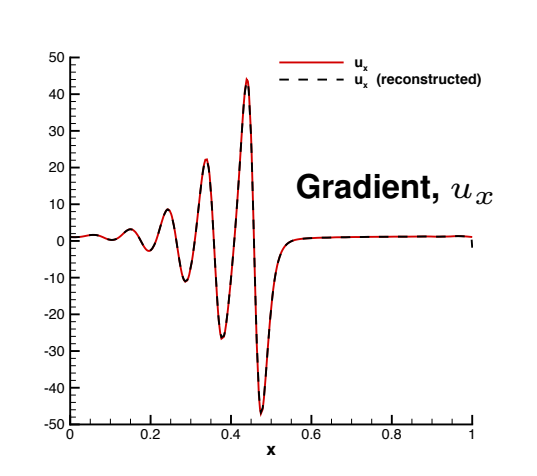


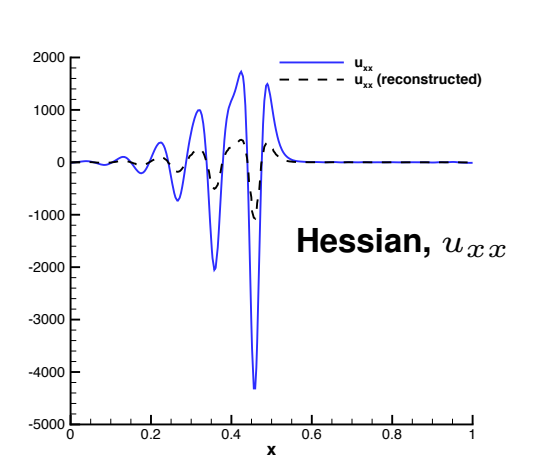




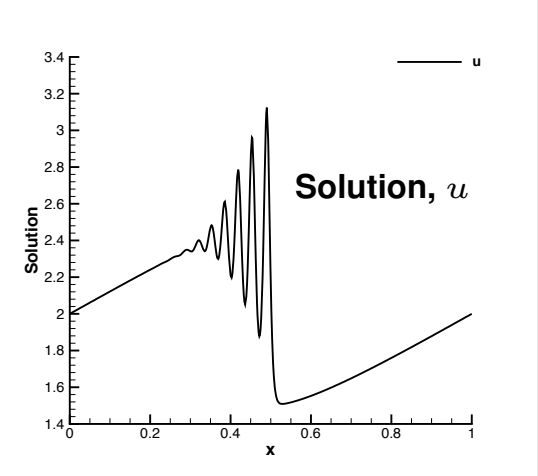
Zero Dispersion Limit: $\epsilon = 10^{-4}$

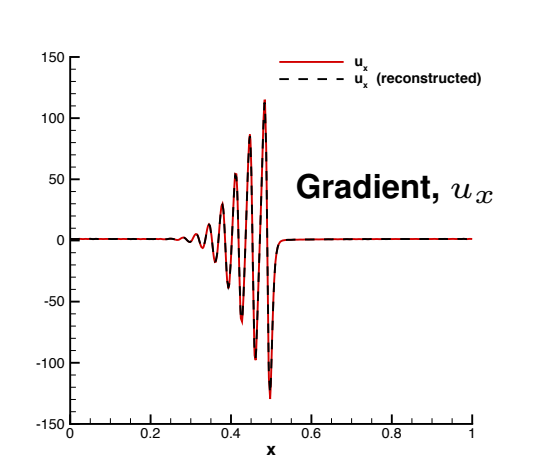


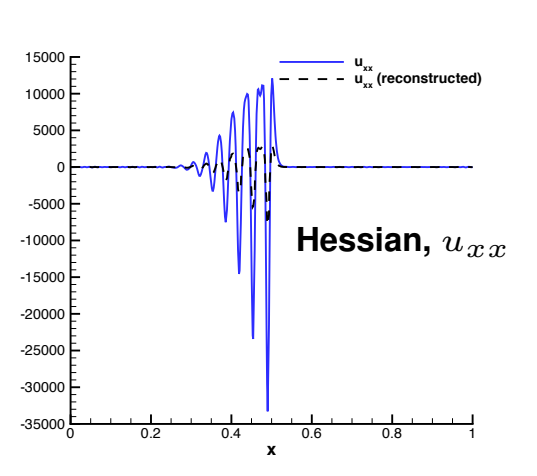




Zero Dispersion Limit: $\epsilon = 10^{-5}$







Summary

- Introduced a NEW first-order system for general Dispersive PDEs
- Showed that the proposed system is hyperbolic (in pseudo-time)
- Illustrated that high-order scheme applied to the proposed system produces equal order of accuracy for solution, gradient and Hessian (second-derivatives)
- Demonstrated the capability of the developed high-order schemes in accurately capturing the dispersive shocks

Next Talk: To be presented in April/May 2016!

- High-Order DG Schemes with the First-Order Hyperbolic System Approach Alireza Mazaheri (NASA) and Hiro Nishikawa (NIA)
  - Same DoF as the conventional DG
  - $(k+2)$-order accurate solution & $(k+1)$-order accurate gradients for Advection problems
  - $(k+1)$-order accurate solution & gradients for Diffusion problems

Acknowledgments

- This work is a work of government and supported internally by NASA Langley Center Chief Technology Office through the Center Innovation Fund (CIF) project