Paper will be available soon.

Accuracy-Preserving Source Term Quadrature for Third-Order Edge-Based Scheme

NATIONAL
INSTITUTE OF
AEROSPACE

Hiroaki Nishikawa and Yi Liu

National Institute of Aerospace

81st NIA CFD Seminar, February 14, 2017

Cradle

Invisible Hand

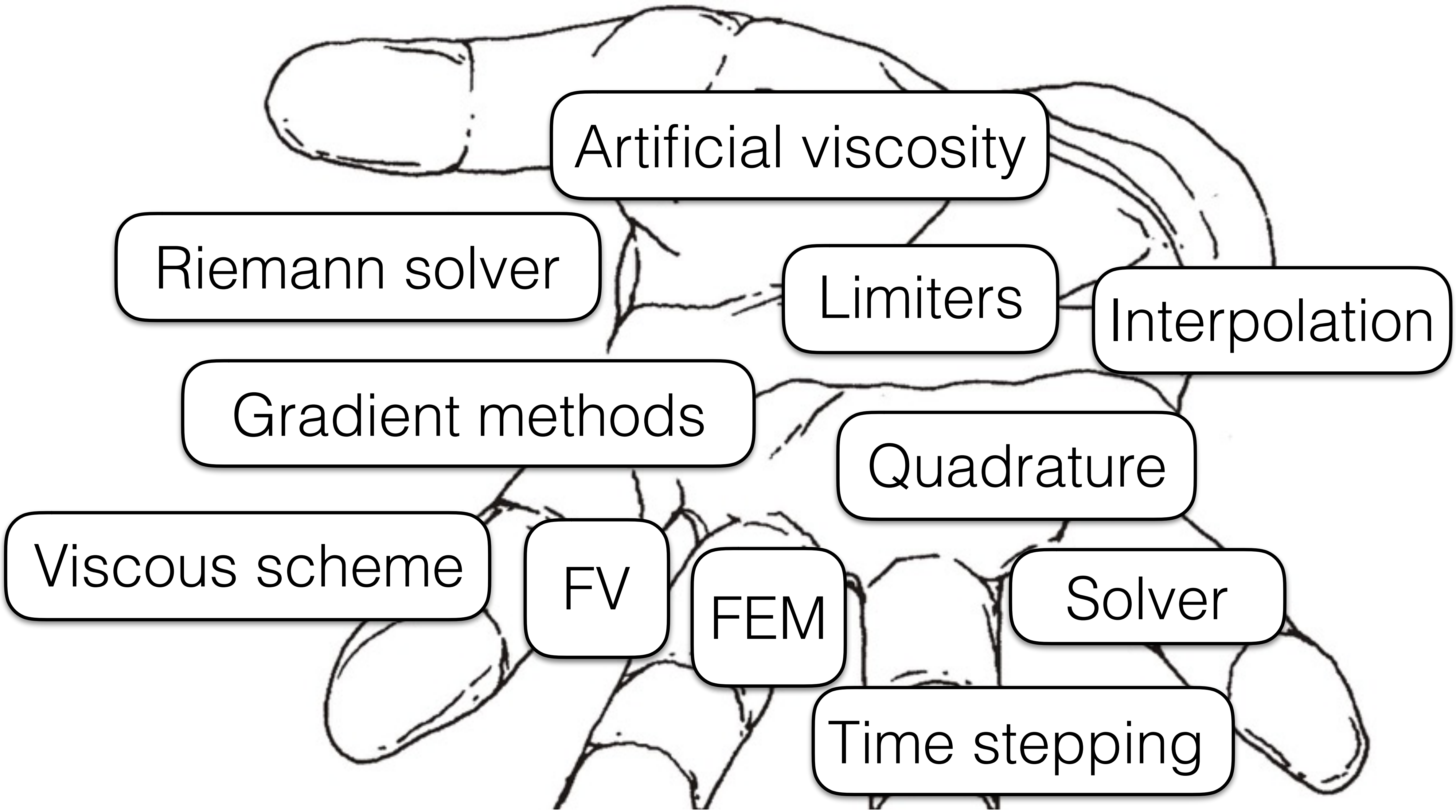
NATIONAL INSTITUTE OF AEROSPACE

"It is not from the benevolence of the butcher, the brewer or the baker that we expect our dinner, but from their regard to their own self-interest...
... led by an invisible hand to promote an end which was no part of his intention. ***By pursuing his own interest, he frequently promotes that of society more effectually*** *than when he really intends to promote it."*

- Adam Smith, The Wealth of Nations, 1776.

The pursuit of self-interest leads to social benefits; individuals should feel free to pursue it.

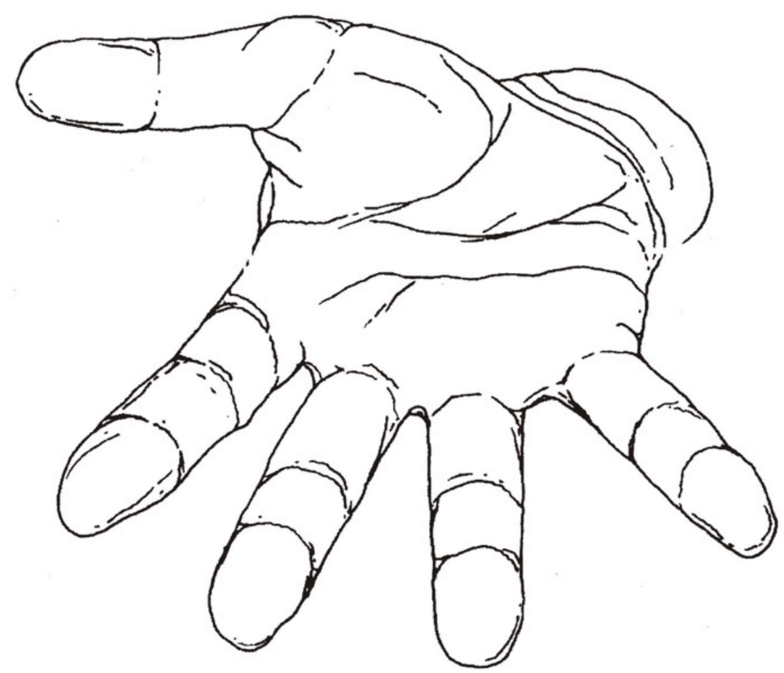
Invisible Hand in CFD



The best algorithm is achieved by individually pursuing the best algorithm in each component.

Third-Order Edge-Based Scheme

Invisible hand?



Edge-Based Scheme

NASA's FUN3D; Software Cradle's SC/Tetra; DLR Tau code, etc.

NATIONAL INSTITUTE OF AEROSPACE

$$\operatorname{div} \mathbf{f} = 0$$

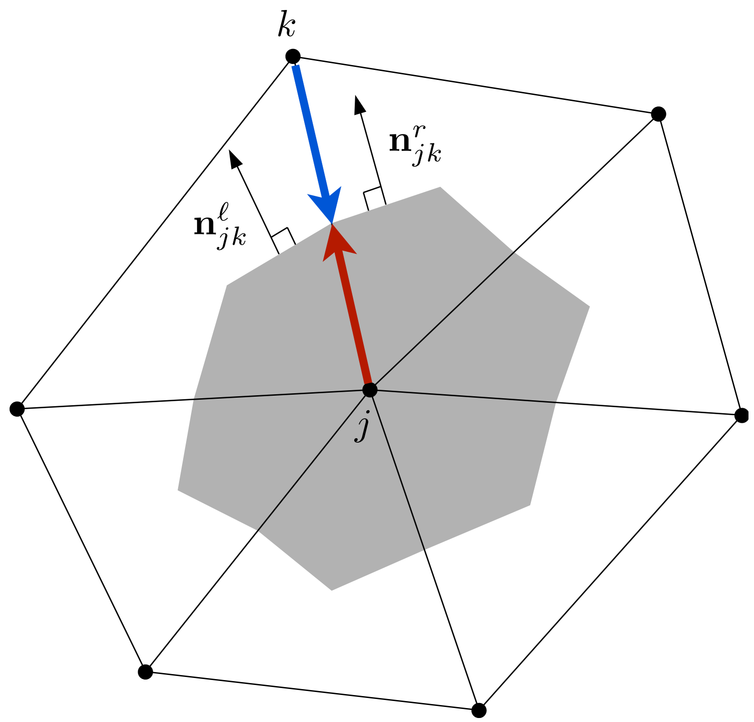
Edge-based discretization:

$$\frac{1}{V_j} \sum_{k\{k_j\}} \phi_{jk} |\mathbf{n}_{jk}| = \operatorname{div}\mathbf{f} + TE$$

Numerical flux at edge-midpoint:

$$\phi_{jk} = \frac{1}{2}(\mathbf{f}_L + \mathbf{f}_R) \cdot \hat{\mathbf{n}}_{jk} - \frac{1}{2}|a_n|(u_R - u_L)$$

$$\mathbf{n}_{jk} = \mathbf{n}_{jk}^{\ell} + \mathbf{n}_{jk}^{r}$$

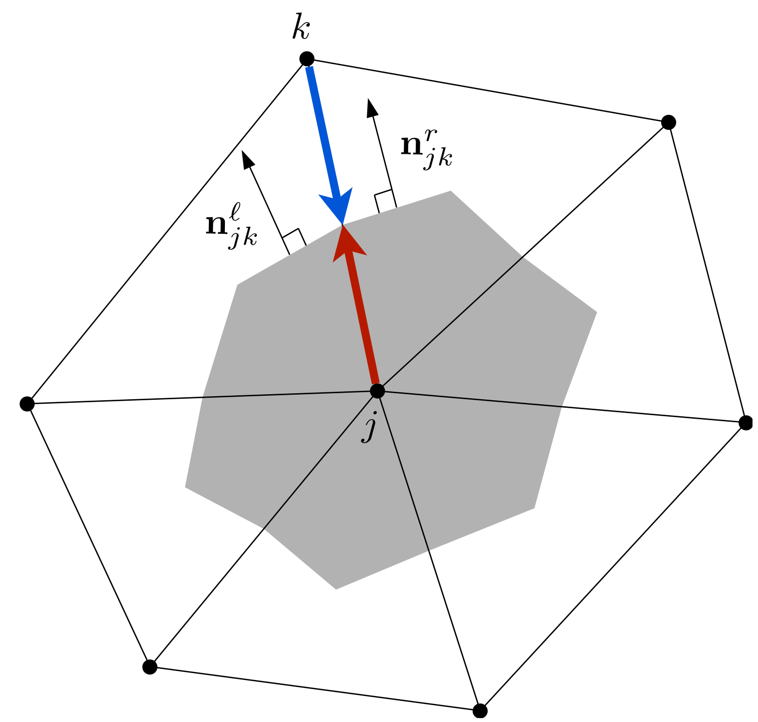


NOTE: EB scheme approximates the differential form in a conservative way.

Third-Order Edge-Based Scheme

NATIONAL INSTITUTE OF AEROSPACE

Arbitrary Triangular/Tetrahedral Grids



2nd-Order (Linear LSQ gradients)

$$u_L = u_j + \frac{1}{2}\overline{\nabla} u_j \cdot \Delta \mathbf{r}_{jk} \qquad \mathbf{f}_L = \mathbf{f}(u_L)$$

$$u_R = u_k - \frac{1}{2}\overline{\nabla} u_k \cdot \Delta \mathbf{r}_{jk} \qquad \mathbf{f}_R = \mathbf{f}(u_R)$$

3rd-Order (Quadratic LSQ gradients) $\overline{\nabla}\mathbf{f}_j = \left(\frac{\partial \mathbf{f}}{\partial u}\right)_j \overline{\nabla} u_j$

$$u_L = u_j + \frac{1}{2}\overline{\nabla} u_j \cdot \Delta \mathbf{r}_{jk} \qquad \mathbf{f}_L = \mathbf{f}_j + \frac{1}{2}\overline{\nabla}\mathbf{f}_j \cdot \Delta \mathbf{r}_{jk}$$

$$u_R = u_k - \frac{1}{2}\overline{\nabla} u_k \cdot \Delta \mathbf{r}_{jk} \qquad \mathbf{f}_R = \mathbf{f}_k - \frac{1}{2}\overline{\nabla}\mathbf{f}_k \cdot \Delta \mathbf{r}_{jk}$$

Katz&Sankaran(JCP2011)

Linear extrapolations

Pursuing the most accurate (quadratic) extrapolation will not bring 3rd-order accuracy to the overall EB discretization.

Nishikawa, JCP2015, AIAA2015-2451

Quadratic LSQ Gradient

Fit a quadratic polynomial over the neighbors + their neighbors.

$$\left(\mathbf{A}^T\mathbf{A}\right)\mathbf{x} = \mathbf{b}$$

Solve it to obtain the solution:

$$\mathbf{x} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{b}$$

and store only the coefficients for the gradient (first three rows of the pseudo-inverse): e.g.,

$$\nabla\rho_j = \sum_{k\in\{k_j\}} \begin{bmatrix} c_{xjk} \\ c_{yjk} \\ c_{zjk} \end{bmatrix} (\rho_k - \rho_j)$$

$$\mathbf{x} = \begin{bmatrix} \partial_x\rho_j \\ \partial_y\rho_j \\ \partial_z\rho_j \\ \partial_{xx}\rho_j \\ \partial_{yy}\rho_j \\ \partial_{zz}\rho_j \\ \partial_{xy}\rho_j \\ \partial_{yz}\rho_j \\ \partial_{zx}\rho_j \end{bmatrix}$$

No need to store the LSQ coefficients for second derivatives.

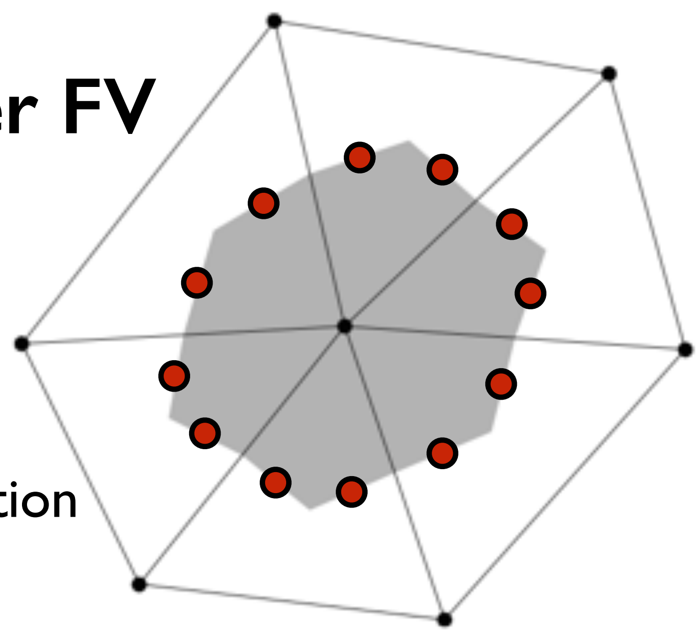
Second-derivatives are not needed.

Finite-Volume vs Edge-Based

Arbitrary elements

2nd-order FV

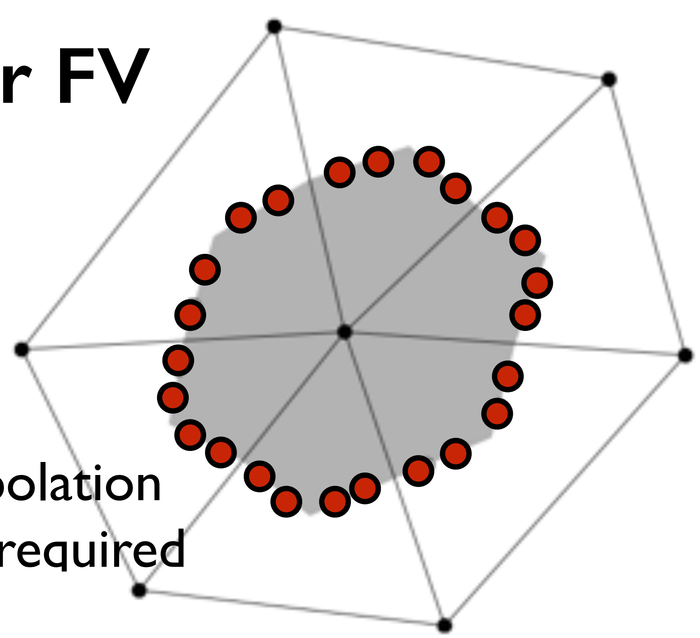
Linear extrapolation



(a) Second-order finite-volume discretization

3rd-order FV

Quadratic extrapolation
second-derivatives required

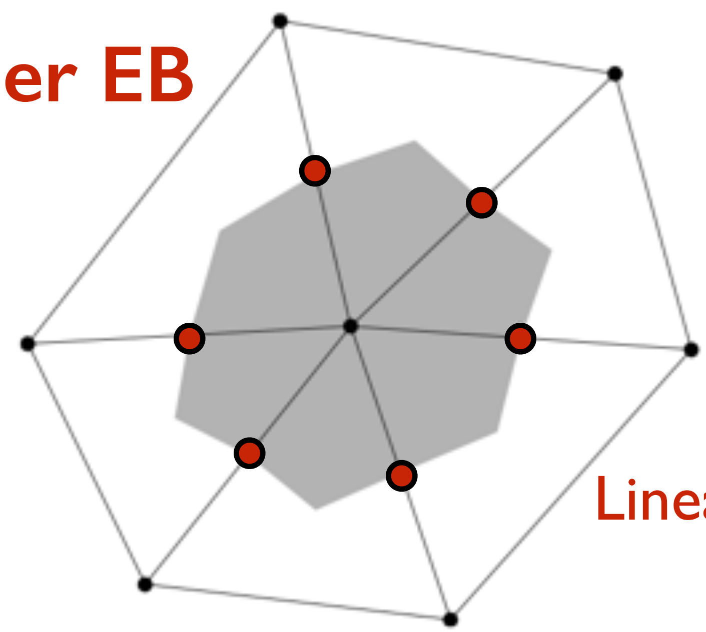


(c) Third-order finite-volume discretization

Simplex elements

2nd-order EB

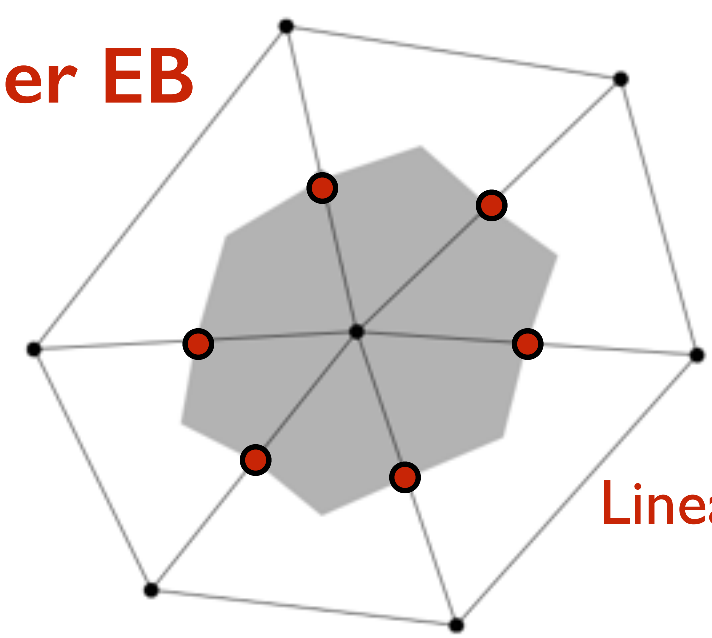
Linear extrapolation



(b) Second-order edge-based discretization

3rd-order EB

Linear extrapolation

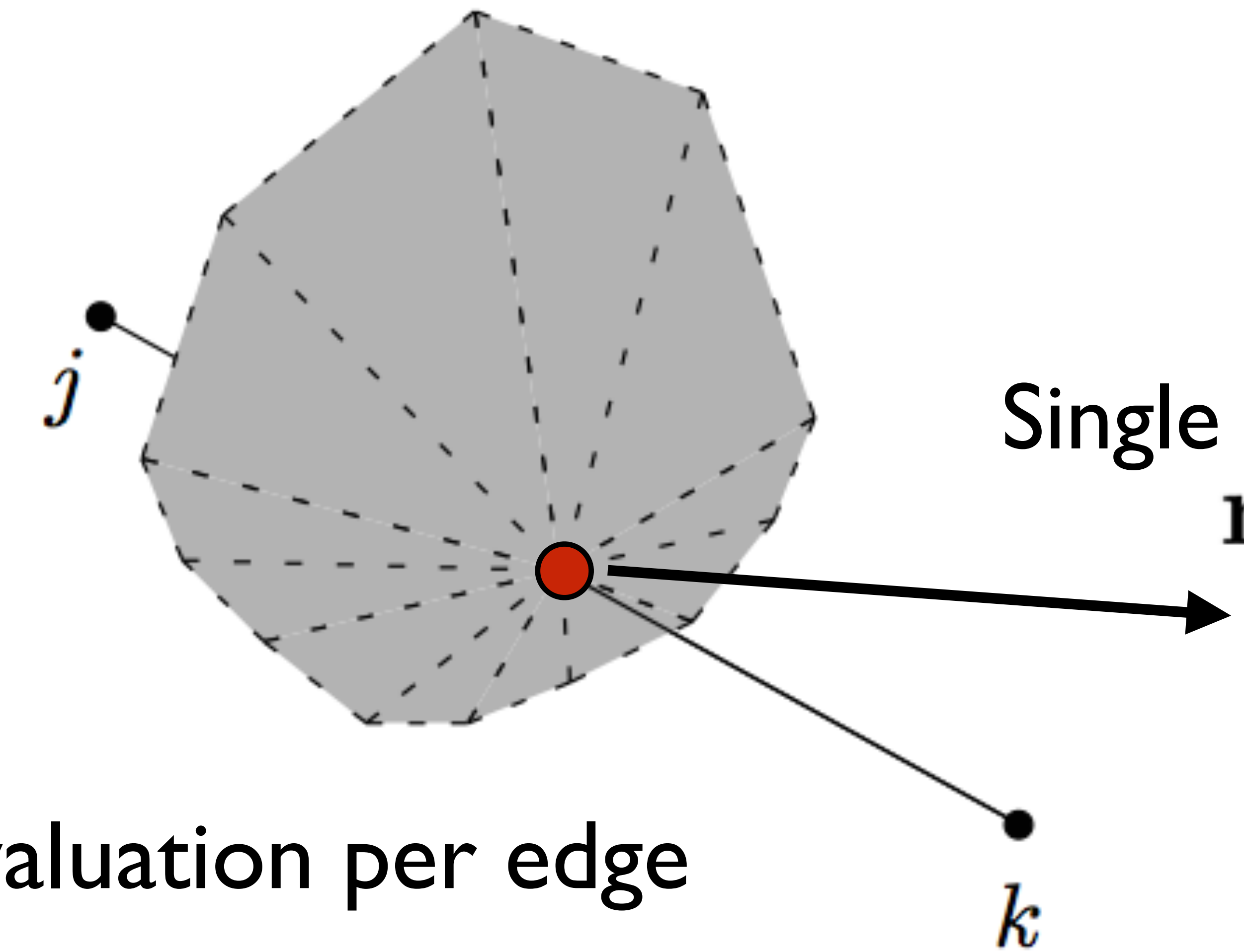


(d) Third-order edge-based discretization

3D Edge-Based Flux Quadrature

NATIONAL INSTITUTE OF AEROSPACE

2nd/3rd-order EB scheme (tetrahedral grids)



Single face normal
$\mathbf{n}_{jk}$

Single flux evaluation per edge

Pursuing a high-order flux quadrature will not bring 3rd-order accuracy to the overall EB discretization.

Third-Order on Linear Mesh

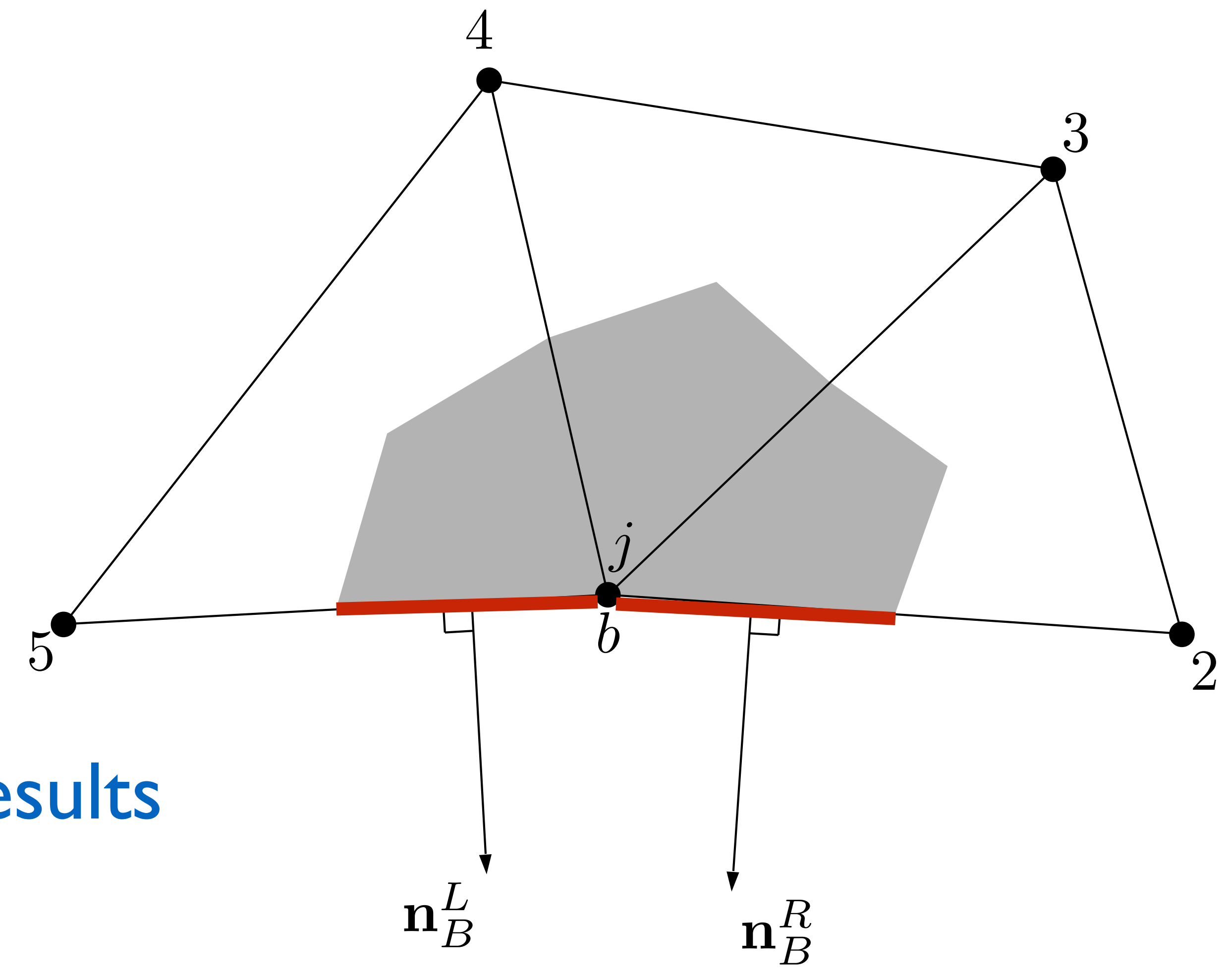
Discretization is exact for quadratic fluxes, even for a curved boundary by a special boundary quadrature.

See JCP2015,

NIA CFD Seminar 12-16-2014

(Presentation file and video)

See AIAA2016-3969 for 3D Euler results



Pursuing a high-order boundary quadrature nor high-order grids will not bring 3rd-order accuracy to the overall EB discretization.

Advection Diffusion (Navier-Stokes)

Accuracy may be lost with your 'best' diffusion scheme:

EB advection + 'Best' diffusion => 2nd-order adv-diff

Successful diffusion schemes:

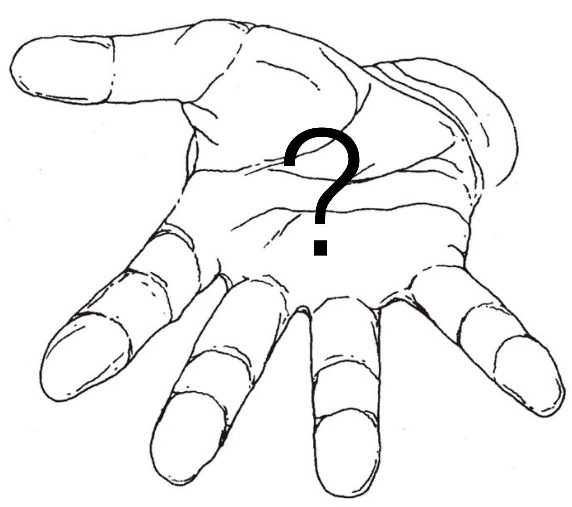
Nishikawa JCP2014, NIA CFD Seminar 06-18-2013, AIAA2015-2451 (NS)

Pincock and Katz, JSC, v61, Issue2, pp454-476 (NS)

Pursuing the best third-order diffusion scheme will not bring 3rd-order accuracy to the overall EB discretization.

Not necessarily the best ones…

- *Edge-based approximation to the flux balance.*
- *Linear extrapolation with quadratic LSQ.*
- *3rd-order with linear grids.*
- *Special boundary flux quadrature.*
- *Special diffusion scheme.*



and now source terms…

Source Term for Third-Order

Source Terms

$$\mathrm{div}\mathbf{f} = s$$

- Reaction terms.
- Method of manufactured solutions.
- Time derivatives by explicit/implicit time stepping (BDF, RK)

$$\frac{1}{V_j} \sum_{k \in \{k_j\}} \phi_{jk} |\mathbf{n}_{jk}| = \frac{1}{V_j} \int_{V_j} s \, dV$$

Pursuing accurate source quadrature -> Point evaluation.

$$\frac{1}{V_j} \int_{V_j} s \, dV = s_j$$

This is the most accurate one.

But then 3rd-order accuracy is lost.

Roles of Government

"According to the system of natural liberty, the sovereign has only three duties to attend to… the duty of protecting the society from the violence… the duty of establishing an exact administration of justice… the duty of erecting and maintaining certain public works…"

- Adam Smith, An Inquiry into the Nature and Causes of the Wealth of Nations(1776).

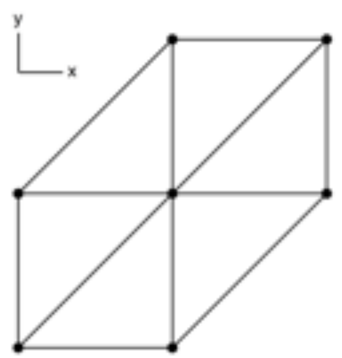
Free market works best,
when protected by laws of justice.

‘Laws of Justice’ for Third-Order

NATIONAL INSTITUTE OF AEROSPACE

Case of no source : $\operatorname{div}\mathbf{f} = 0$

(1) Regular grids: 2nd-order TE vanishes.



$$\frac{1}{V_j}\sum_{k\in\{k_j\}}\phi_{jk}(\mathbf{n}_{jk}) = \operatorname{div}\mathbf{f}_j - \frac{1}{24V_j}\left[Q_{xx}+Q_{yy}+Q_{xy}\right](\operatorname{div}\mathbf{f})_j + O(h^3).$$

$$Q_{xx} = \sum_{k\in\{k_j\}} n_x\Delta x^3\partial_{xx},\quad Q_{yy} = \sum_{k\in\{k_j\}} n_y\Delta y^3\partial_{yy},\quad Q_{xy} = \sum_{k\in\{k_j\}} 3n_x\Delta x^2\Delta y\,\partial_{xy}$$

(2) Irregular grids: 1st-order TE is not generated.

$$\frac{1}{V_j}\sum_{k\in\{k_j\}}\phi_{jk}(\mathbf{n}_{jk}) = \operatorname{div}\mathbf{f}_j + Ch^2$$

Either one of them alone does not guarantee 3rd-order accuracy on arbitrary grids.

Any additional term in PDE must be discretized under these conditions, to protect the 3rd-order accuracy.

Case of CL with Source Term

(1) Regular grids: 2nd-order TE must vanish.

$$\frac{1}{V_j}\sum_{k\in\{k_j\}}\phi_{jk}|\mathbf{n}_{jk}| - \frac{1}{V_j}\int_{V_j} s\,dV = \mathrm{div}\mathbf{f} - s - \frac{1}{24}\left[Q_{xx}+Q_{xy}+Q_{yy}\right](\mathrm{div}\mathbf{f} - s) + O(h^3)$$

Point source: $\frac{1}{V_j}\sum_{k\in\{k_j\}}\phi_{jk}|\mathbf{n}_{jk}| - \frac{1}{V_j}\int_{V_j} s\,dV = \mathrm{div}\mathbf{f} - s - \frac{1}{24}\left[Q_{xx}+Q_{yy}+Q_{xy}\right]\underline{\underline{\mathrm{div}\mathbf{f}}} + O(h^3)$

(2) Irregular grids: 1st-order TE must not be generated.

$$\frac{1}{V_j}\sum_{k\in\{k_j\}}\phi_{jk}|\mathbf{n}_{jk}| - \frac{1}{V_j}\int_{V_j} s\,dV = \mathrm{div}\mathbf{f} - s + O(h^2)$$

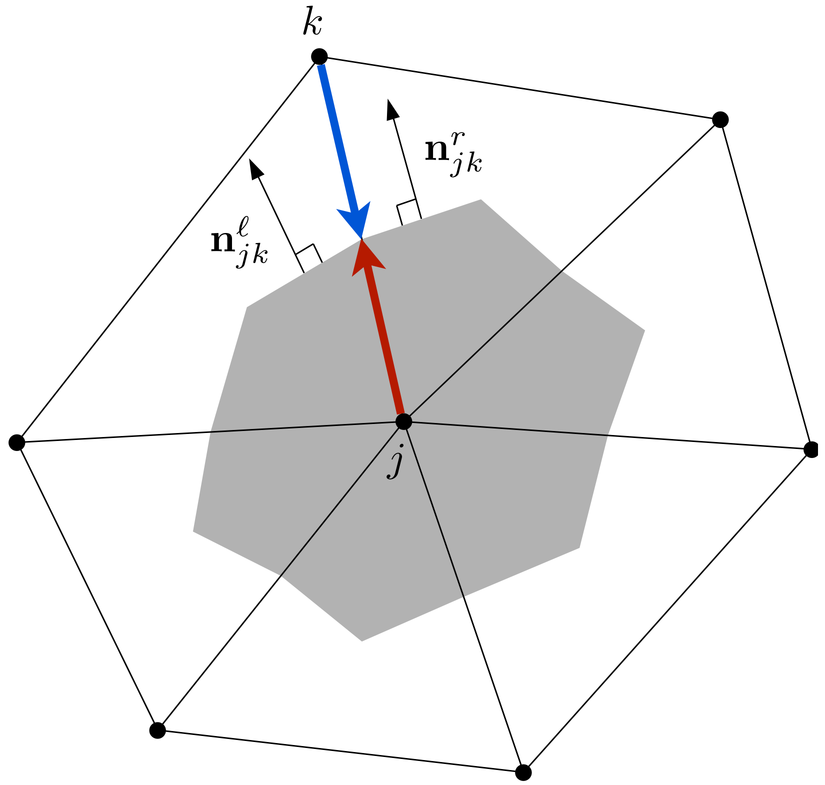
Source quadrature formulas must satisfy these two conditions.

Source quadrature must have a 2nd-order TE.

That's why the best source discretization doesn't work.

Previously Known Formulas (1)

Katz formula (2011):



$$\int_{V_j} s\, dV = \sum_{k \in \{k_j\}} \frac{1}{2}(s_L + s_R) V_{jk}$$

$$s_L = s_j - \frac{1}{2}\hat{\partial}_{jk} s_j - \frac{1}{8}\hat{\partial}^2_{jk} s_j, \quad s_R = s_k - \frac{1}{2}\hat{\partial}_{jk} s_k - \frac{1}{8}\hat{\partial}^2_{jk} s_k$$

$$V_{jk} = \frac{1}{4}(\Delta \mathbf{x}_{jk} \cdot \mathbf{n}_{jk})$$

$$\hat{\partial}_{jk} \equiv \Delta \mathbf{x}_{jk} \cdot (\hat{\partial}_x, \hat{\partial}_y)$$

Require second-derivatives of source terms…

Previously Known Formulas (2)

NATIONAL INSTITUTE OF AEROSPACE

Divergence formulation: $s = \partial_x f^s + \partial_y g^s$

See Nishiakwa, JCP2012

Symmetric: Div(sym)

$$\begin{aligned} f^s &= \frac{1}{2}(x-x_j)s - \frac{1}{4}(x-x_j)^2\partial_x s + \frac{1}{12}(x-x_j)^3\partial_{xx}s, \\ g^s &= \frac{1}{2}(y-y_j)s - \frac{1}{4}(y-y_j)^2\partial_y s + \frac{1}{12}(y-y_j)^3\partial_{yy}s. \end{aligned}$$

$$\downarrow$$

$$\partial_x f^s + \partial_y g^s = s + \underbrace{\frac{1}{12}(x-x_j)^3\partial_{xxx}s + \frac{1}{12}(y-y_j)^3\partial_{yyy}s}_{O(h^3)}$$

One-component : Div(x)

$$\begin{aligned} f^s &= (x-x_j)s - \frac{1}{2}(x-x_j)^2\partial_x s + \frac{1}{6}(x-x_j)^3\partial_{xx}s \\ g^s &= 0, \end{aligned}$$

$$\downarrow$$

$$\partial_x f^s + \partial_y g^s = s + \underbrace{\frac{1}{6}(y-y_j)^3\partial_{yyy}s}_{O(h^3)}$$

Special Divergence form ($\mathrm{curl}\,\mathbf{s} = 0$):

See AIAA2015-2451

$$\mathbf{f}^s = \frac{\mathbf{s}\otimes\Delta\mathbf{x} - (\mathbf{s}\cdot\Delta\mathbf{x})\mathbf{I}}{2} \longrightarrow \mathbf{s} = \mathrm{div}\mathbf{f}^s$$

Straightforward discretization by 3rd-order EB scheme.

Problem: Second Derivatives

NATIONAL INSTITUTE OF AEROSPACE

Second derivatives of source terms are required.

E.g., 3D unsteady NS:

6 second-derivatives for 5 variables must be computed and stored at every backplane.

$$\partial_{xx}\rho, \partial_{yy}\rho, \partial_{zz}\rho, \partial_{xy}\rho, \partial_{yz}\rho, \partial_{zx}\rho, \ldots..$$

Objective:

Remove 2nd derivatives from source quadrature, and develop a 3rd-order unstructured method that does not require 2nd-derivatives at all.

Accuracy-Preserving Quadrature

Seek a formula in the form:

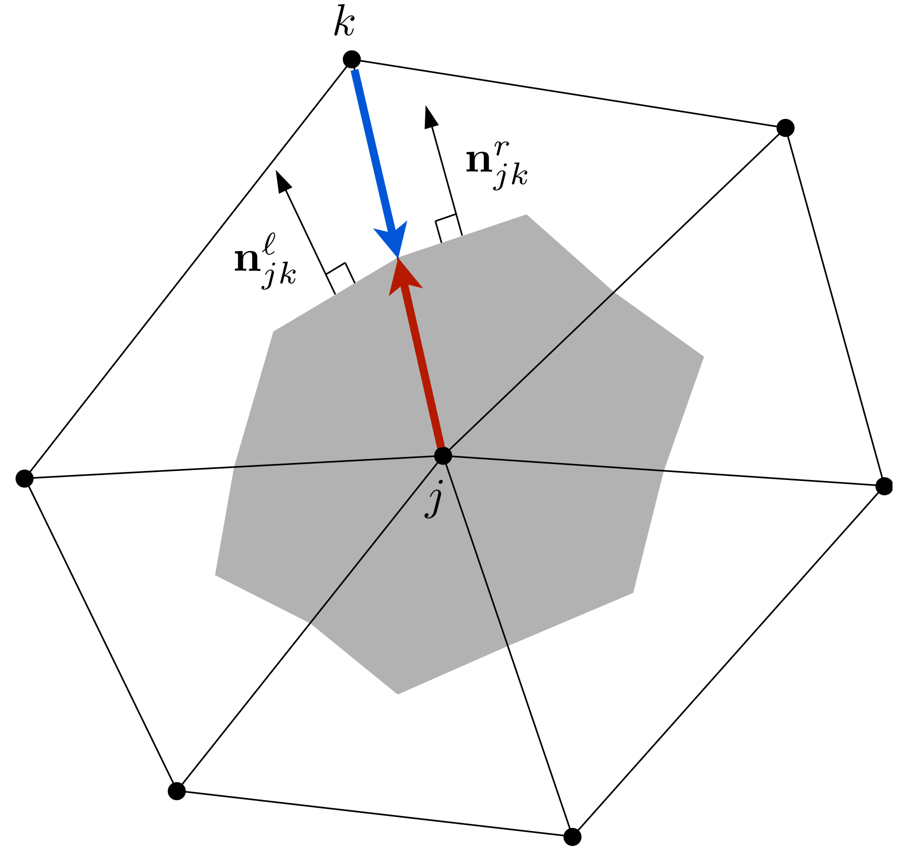
$$\int_{V_j} s\, dV = \sum_{k \in \{k_j\}} \frac{1}{2}(s_L + s_R) V_{jk}$$

$$s_L = \underline{a_L} s_j + \underline{b_L} \hat{\partial}_{jk} s_j + \underline{c_L} \hat{\partial}_{jk}^2 s_j,$$

$$s_R = \underline{a_R} s_k + \underline{b_R} \hat{\partial}_{jk} s_k + \underline{c_R} \hat{\partial}_{jk}^2 s_k$$

$$\hat{\partial}_{jk} \equiv \Delta \mathbf{x}_{jk} \cdot (\hat{\partial}_x, \hat{\partial}_y) \qquad V_{jk} = \frac{1}{4}(\Delta \mathbf{x}_{jk} \cdot \mathbf{n}_{jk})$$

$$\mathbf{n}_{jk} = \mathbf{n}_{jk}^{\ell} + \mathbf{n}_{jk}^{r}$$



Determine $a_L, b_L, c_L, a_R, b_R, c_R$ to achieve third-order.

Two conditions from Irregular grids

NATIONAL INSTITUTE OF AEROSPACE

Taylor expansion up to 2nd-order:

This vanishes on regular grids.

$$\sum_{k \in \{k_j\}} \frac{1}{2}(s_L + s_R) V_{jk} =$$

$$\frac{a_L + a_R}{2} s_j V_j + \frac{a_R + b_L + b_R}{2} \sum_{k \in \{k_j\}} \partial_{jk} s_j V_{jk} + \frac{2(b_R + c_R + c_L) + a_R}{4} \sum_{k \in \{k_j\}} \partial_{jk}^2 s_j V_{jk}$$

(1) Consistency: $\frac{a_L + a_R}{2} = 1$

This condition is not needed for regular grids.

(2) Eliminate 1st-order error: $\frac{a_R + b_L + b_R}{2} = 0$

Two conditions must be satisfied, but not enough.

One condition from regular grids

NATIONAL INSTITUTE OF AEROSPACE

Taylor expansion on regular grids:

$$\frac{1}{V_j}\sum_{k\in\{k_j\}}\frac{1}{2}(s_L+s_R)V_{jk} = s_j + \frac{2(b_R+c_R+c_L)+a_R}{4}\left(\frac{1}{3V_j}\right)\left[Q_{xx}+Q_{yy}+Q_{xy}\right]s_j$$

$$\frac{1}{V_j}\sum_{k\in\{k_j\}}\phi_{jk}(\mathbf{n}_{jk}) \quad = \quad \operatorname{div}\mathbf{f}_j - \frac{1}{24V_j}\left[Q_{xx}+Q_{yy}+Q_{xy}\right](\operatorname{div}\mathbf{f})_j + O(h^3)$$

(3) Compatibility: $\dfrac{2(b_R+c_R+c_L)+a_R}{4}\left(\dfrac{1}{3}\right) = -\dfrac{1}{24}$

$$\frac{1}{V_j}\sum_{k\in\{k_j\}}\phi_{jk}(\mathbf{n}_{jk}) - \sum_{k\in\{k_j\}}\frac{1}{2}(s_L+s_R)V_{jk}$$

$$= \underline{\operatorname{div}\mathbf{f}_j - s_j} - \frac{1}{24V_j}\left[Q_{xx}+Q_{yy}+Q_{xy}\right]\left(\underline{\operatorname{div}\mathbf{f}_j - s_j}\right) + O(h^3)$$

so that 2nd-order error vanishes.

Arbitrary Dimensions

NATIONAL INSTITUTE OF AEROSPACE

(1) Consistency: $\frac{a_L + a_R}{2} = 1$

(2) Eliminate 1st-order error: $\frac{a_R + b_L + b_R}{2} = 0$

(3) Compatibility: $\frac{2(b_R + c_R + c_L) + a_R}{4} \left(\frac{D+2}{6D} \right) = -\frac{1}{24}$

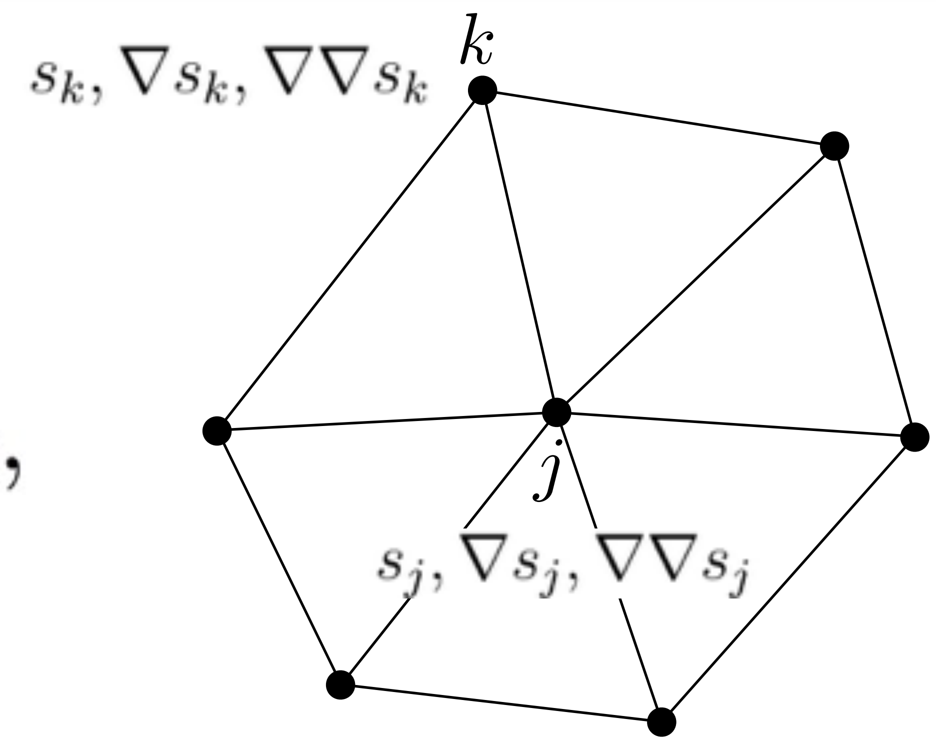
D = 1, 2, 3

3 equations for 6 unknowns: Infinitely many solutions.

$a_L, b_L, c_L, a_R, b_R, c_R$

We now pursue our self-interest for source term by adding three more conditions.

1. Symmetric Formulas



Symmetry conditions:

$$a_L = a_R, \quad b_L = b_R, \quad c_L = c_R,$$

gives

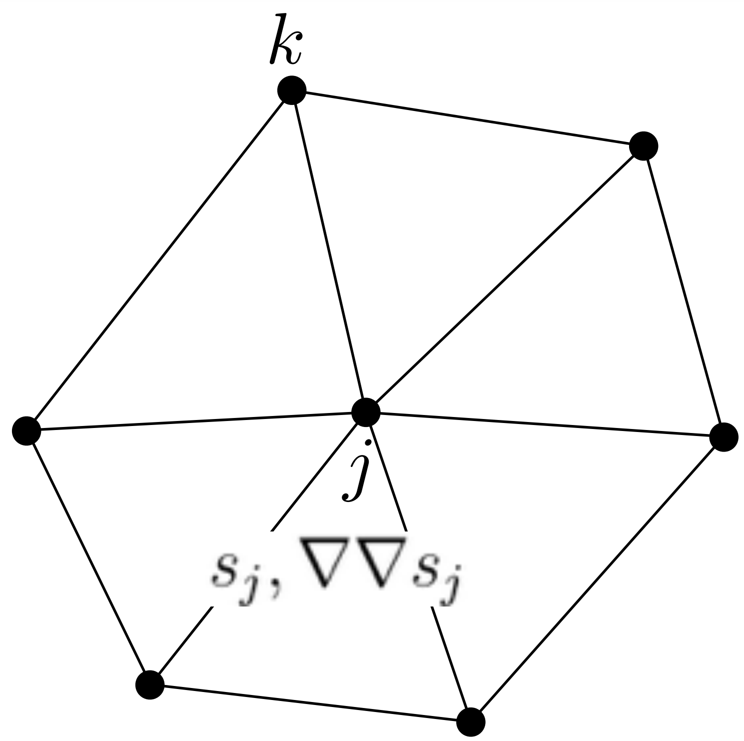
$$\boxed{a_L = a_R = 1, \quad b_L = b_R = -\frac{1}{2}, \quad c_L = c_R = -\frac{D}{4(D+2)}}$$

Equivalent to Katz formula for D=2 (different for D=1 and 3).

2. One-Sided Formula

No contribution from the neighbor:

$$a_R = b_R = c_R = 0$$



gives

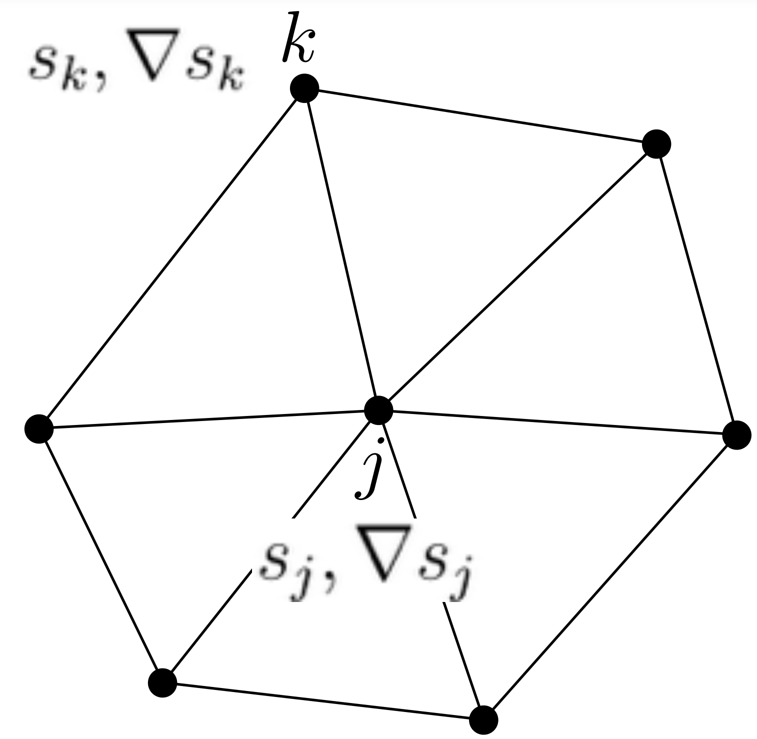
$$\boxed{a_L = 2, \quad b_L = 0, \quad c_L = -\frac{D}{2(D+2)}, \quad a_R = b_R = c_R = 0}$$

This requires second-derivatives at node j.

3. Economical Formulas

No second derivatives:

$$a_L = 1, \quad c_L = c_R = 0$$



gives

$$\boxed{a_L = a_R = 1, \quad b_L = -\frac{1}{D+2}, \quad b_R = -\frac{D+1}{D+2}, \quad c_L = c_R = 0.}$$

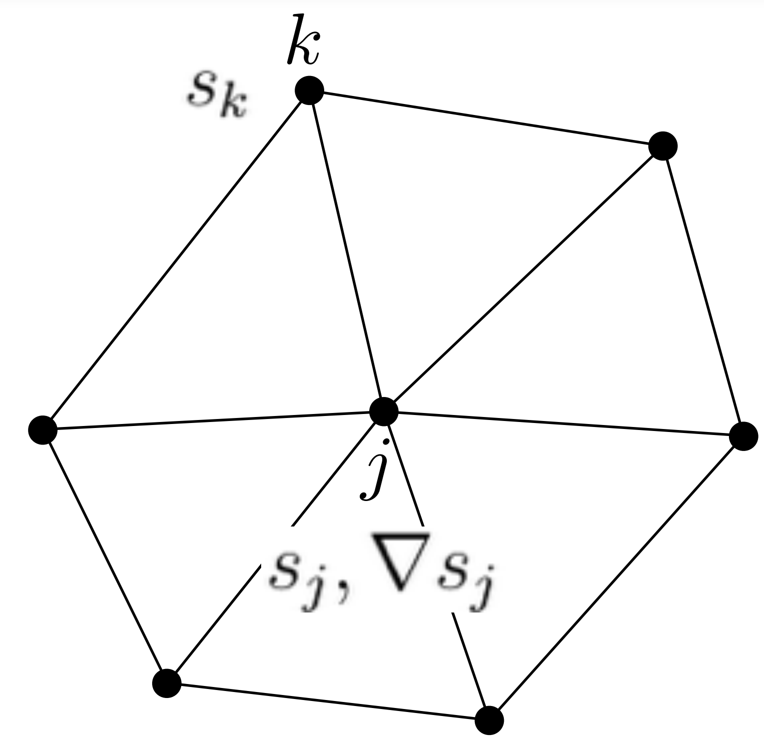
Second derivatives are not needed.
This formula is not unique, allowing a free parameter.

4. Compact Formula

No derivatives at neighbor nodes:

$$b_R = c_L = c_R = 0$$

gives



$$\boxed{a_L = \frac{3D+4}{D+2}, \quad a_R = -\frac{D}{D+2}, \quad b_L = \frac{D}{D+2}, \quad b_R = c_L = c_R = 0.}$$

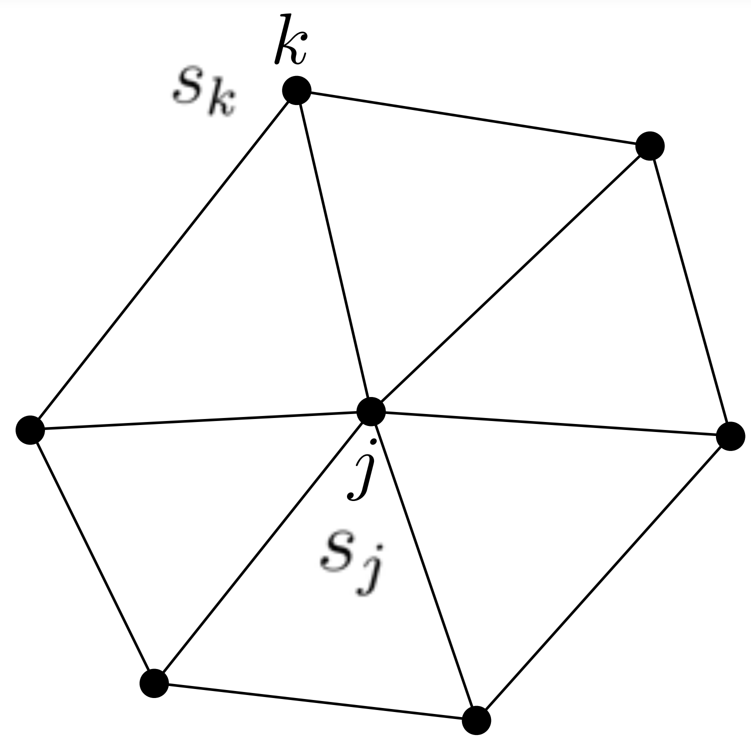
Unique formula: depends only on sj, grad(s)j, and sk.

Compact stencil if grad(s)j is compact.

5. Regular Formula

NATIONAL INSTITUTE OF AEROSPACE

O(h) TE error identically vanishes.
Condition(2) is redundant.
We can impose 4 conditions:



$$b_L = b_R = c_L = c_R = 0$$

which gives

$$\boxed{a_L = \frac{3D+4}{D+2}, \quad a_R = -\frac{D}{D+2}, \quad b_L = b_R = c_L = c_R = 0.}$$

Compare with Compact formula:

$$a_L = \frac{3D+4}{D+2}, \quad a_R = -\frac{D}{D+2}, \quad b_L = \frac{D}{D+2}, \quad b_R = c_L = c_R = 0.$$

Derived Formulas

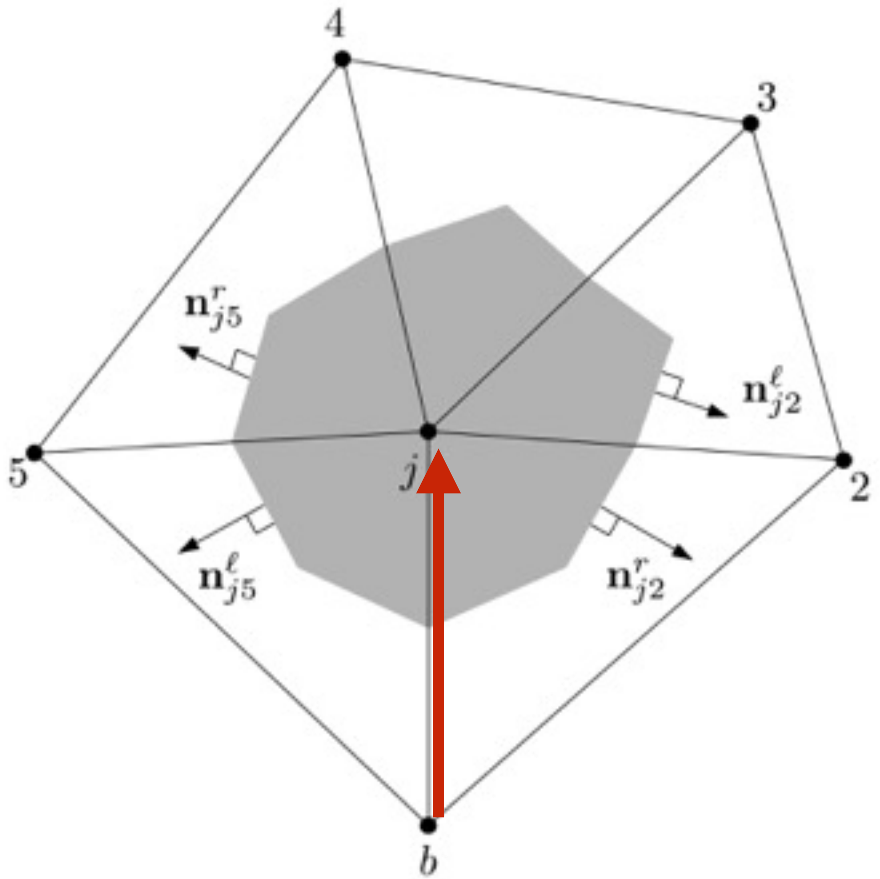
	Grid type	a_L	b_L	c_L	a_R	b_R	c_R
Regular	Regular simplex	$\frac{3D+4}{D+2}$	0	0	$-\frac{D}{D+2}$	0	0
Compact	Arbitrary simplex	$\frac{3D+4}{D+2}$	$\frac{D}{D+2}$	0	$-\frac{D}{D+2}$	0	0
Economical(1)	Arbitrary simplex	1	$-\frac{1}{D+2}$	0	1	$-\frac{D+1}{D+2}$	0
One-sided	Arbitrary simplex	2	0	$-\frac{D}{2(D+2)}$	0	0	0
Symmetric	Arbitrary simplex	1	$-\frac{1}{2}$	$-\frac{D}{4(D+2)}$	1	$-\frac{1}{2}$	$-\frac{D}{4(D+2)}$

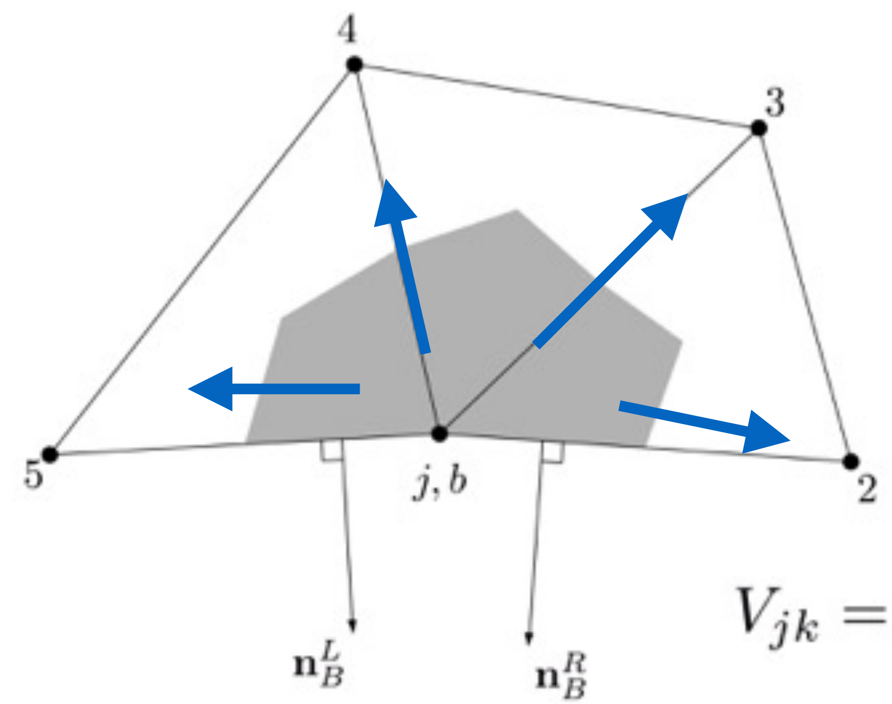
Table 1: Summary of accuracy-preserving source term quadrature formulas. $D = 2$ for triangular grids, and $D = 3$ for tetrahedral grids (and $D = 1$ for one-dimensional grids). Top one is for the regular grid, the next two do not require second derivatives of the source term; the bottom two require the second derivatives. The value in the parenthesis for Economical indicates the chosen value of a_L to generate the formula from the one-parameter family of formulas satisfying $c_L = c_R = 0$.

At Boundary nodes

Edge-collapsing to create a boundary stencil: See JCP2015,

NIA CFD Seminar 12-16-2014
(Presentation file and video)





$$V_{jk} = \frac{1}{4}(\Delta \mathbf{x}_{jk} \cdot \mathbf{n}_{jk})$$

$$\int_{V_j} s\,dV = \sum_{k=3}^{4} \frac{1}{2} s_{jk} V_{jk} + \frac{1}{2} s_{j2} V_{j2}(\mathbf{n}^\ell_{j2}) + \cancel{\frac{1}{2} s_{j2} V_{j2}(\mathbf{n}^r_{j2})} + \cancel{\frac{1}{2} s_{j5} V_{j5}(\mathbf{n}^\ell_{j5})} + \frac{1}{2} s_{j5} V_{j5}(\mathbf{n}^r_{j5}) + \cancel{\frac{1}{2} s_{jb} V_{jb}}$$

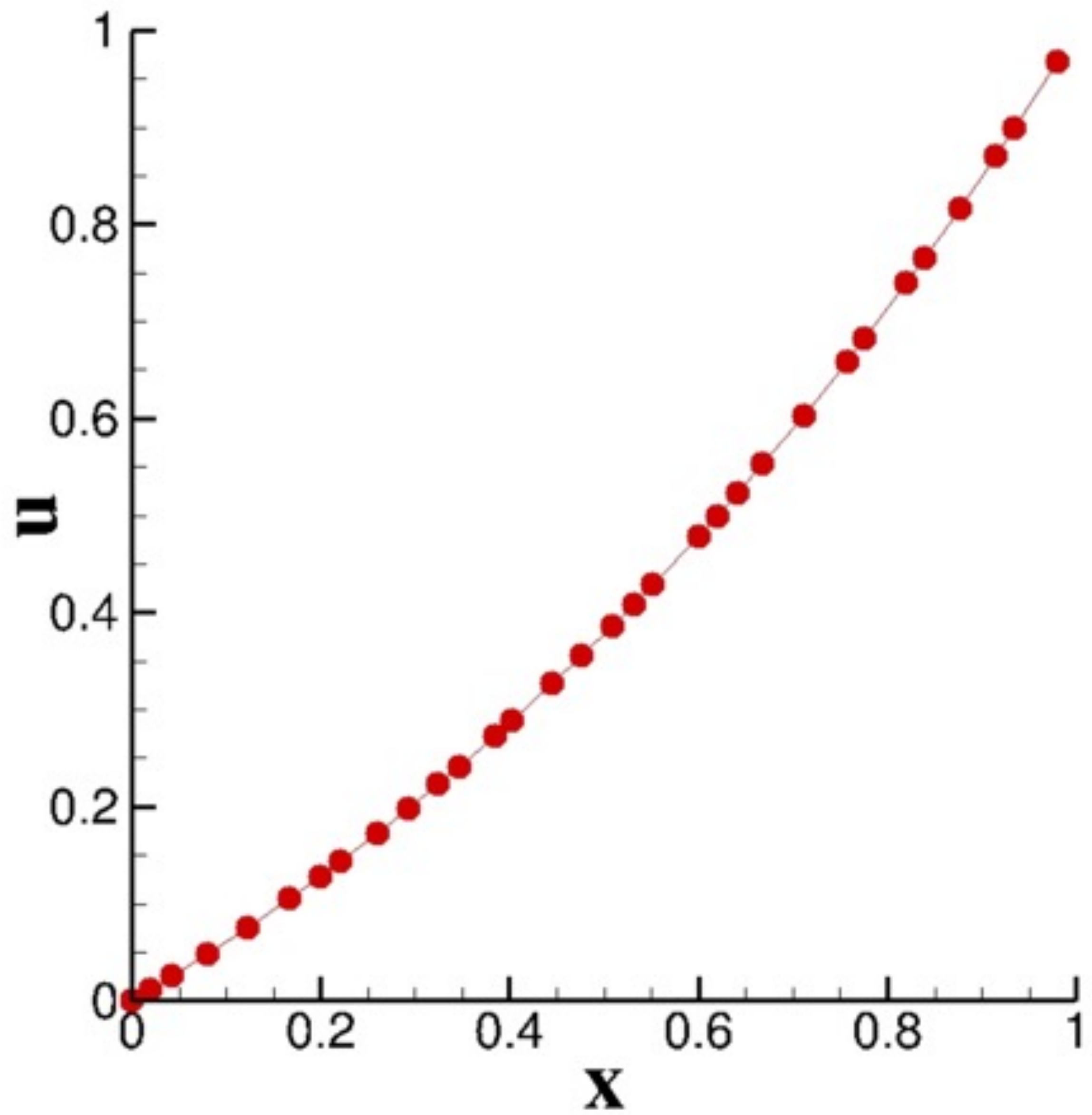
$$\int_{V_j} s\,dV = \sum_{k=3}^{4} \frac{1}{2} s_{jk} V_{jk} + \frac{1}{2} s_{j2} V_{j2}(\mathbf{n}^\ell_{j2}) + \frac{1}{2} s_{j5} V_{j5}(\mathbf{n}^r_{j5}).$$

No boundary closure is required: a single loop over edges suffices.

1D Results

NATIONAL INSTITUTE OF AEROSPACE

Advection equation with a source term:



$$\partial_x u = s(x)$$

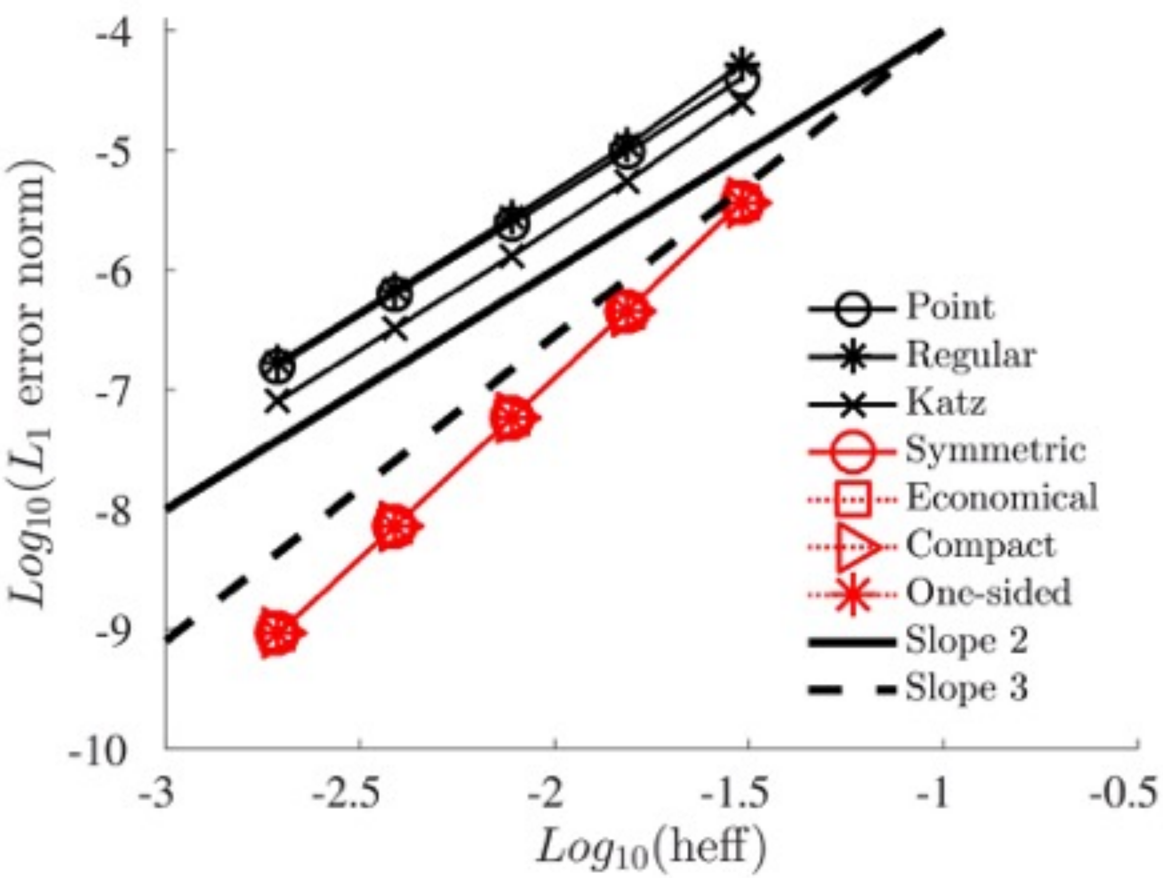
$$s(x) = -\exp(x)/(1 - \exp(1))$$

Exact solution:

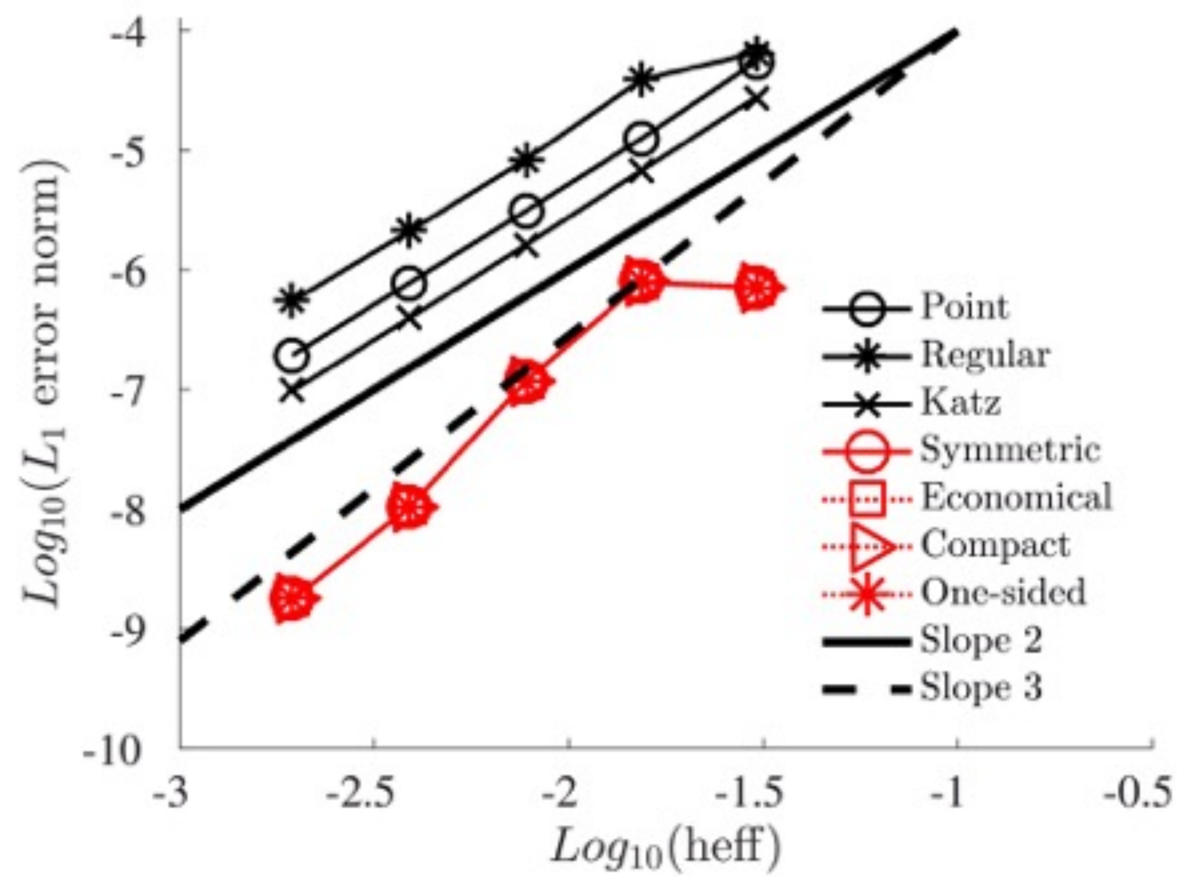
$$u(x) = \frac{1 - \exp(x)}{1 - \exp(1)}$$

Irregular grids:
31, 63, 127, 255, 511 nodes

1D Results



(a) Interior nodes.

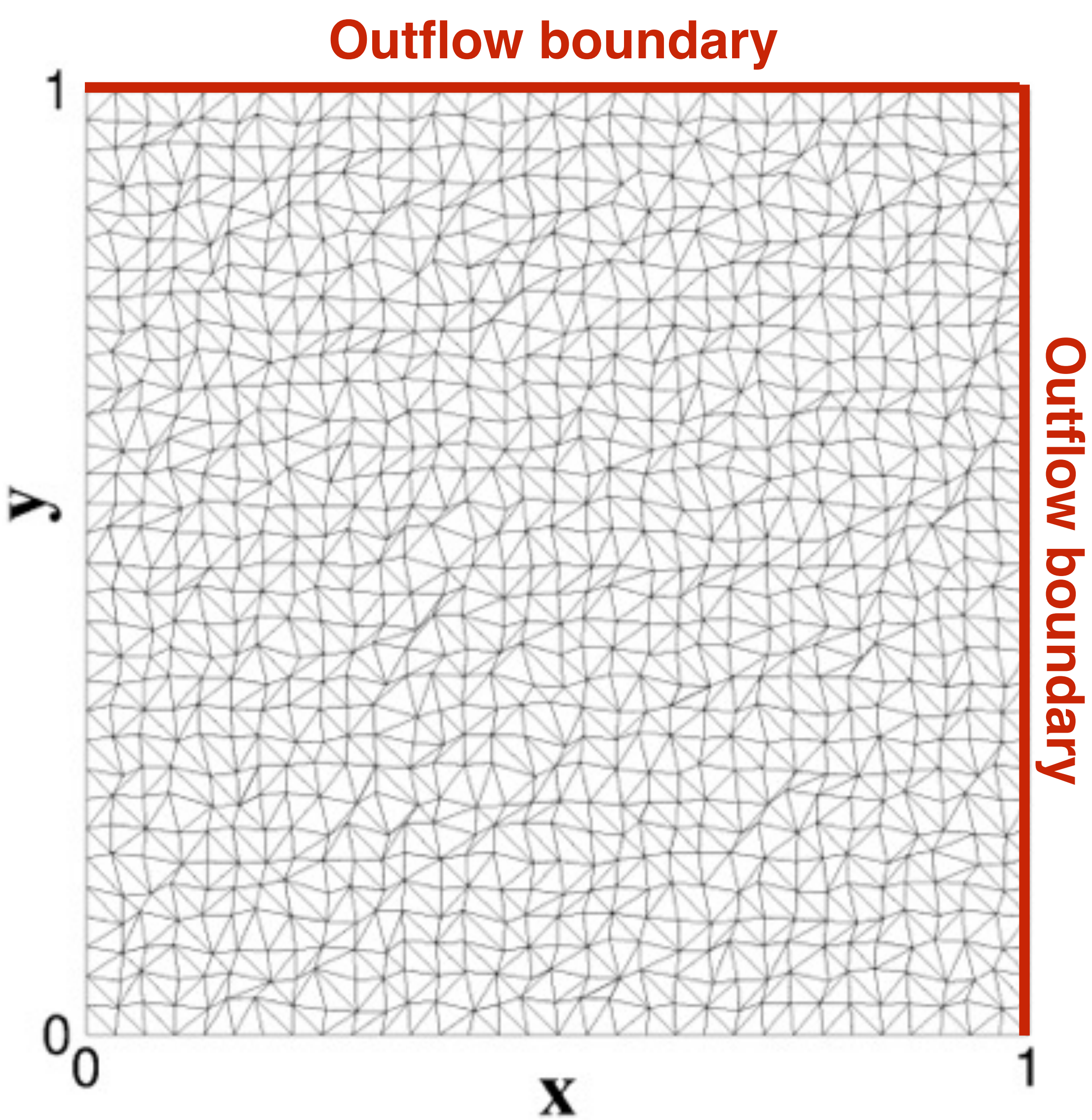


(b) Outflow boundary node ($x = 1$).

All new formulas with D=1 (red) yield 3rd-order accuracy.

2D Results

Burgers' equation with a source term:



$$\partial_x(u^2/2) + \partial_y u = s$$

$$s = \cos(x-y)\sin(x-y) - \cos(x-y)$$

Exact solution:

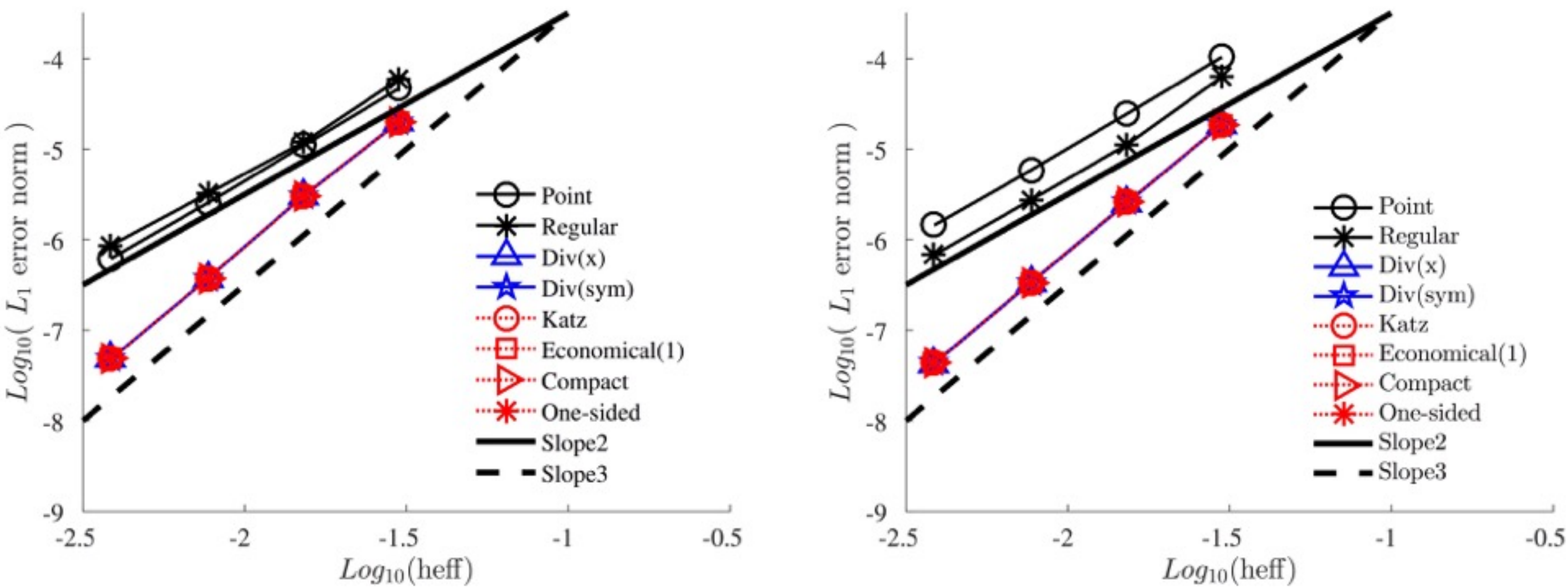
$$u(x,y) = 2 + \sin(x-y)$$

Irregular grids: nxn nodes,
n = 33, 65, 129, 257

2D Results

NATIONAL INSTITUTE OF AEROSPACE

Burgers' equation, irregular grids:



(a) Interior nodes.

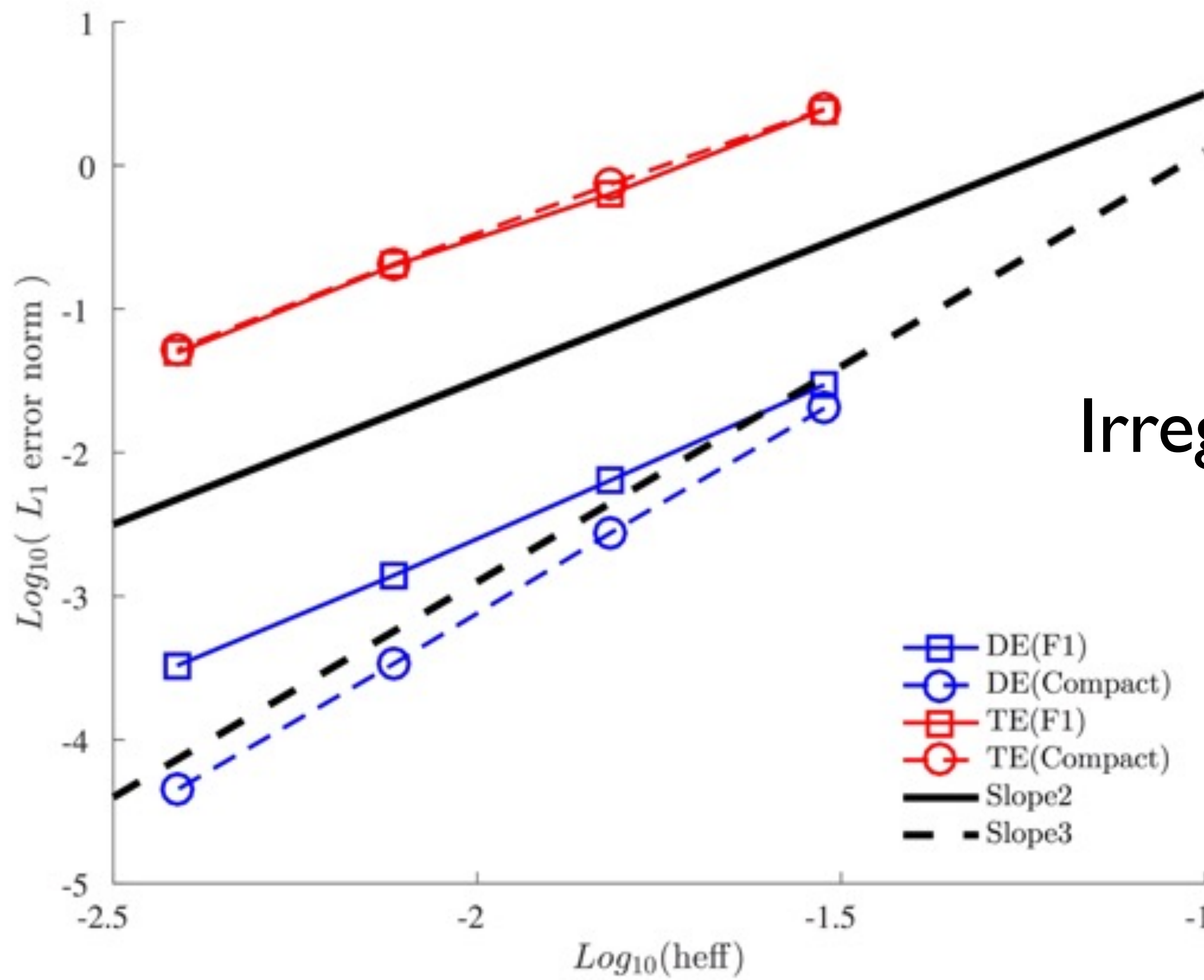
(b) Outflow boundary nodes.

All formulas give 3rd-order except Point and Regular.

2D: Incompatible formula

Consistent and vanishing 1st-order error, but not compatible:

F1: $a_L = a_R = 1,\ b_L = -1,\ b_R = c_L = cR = 0$

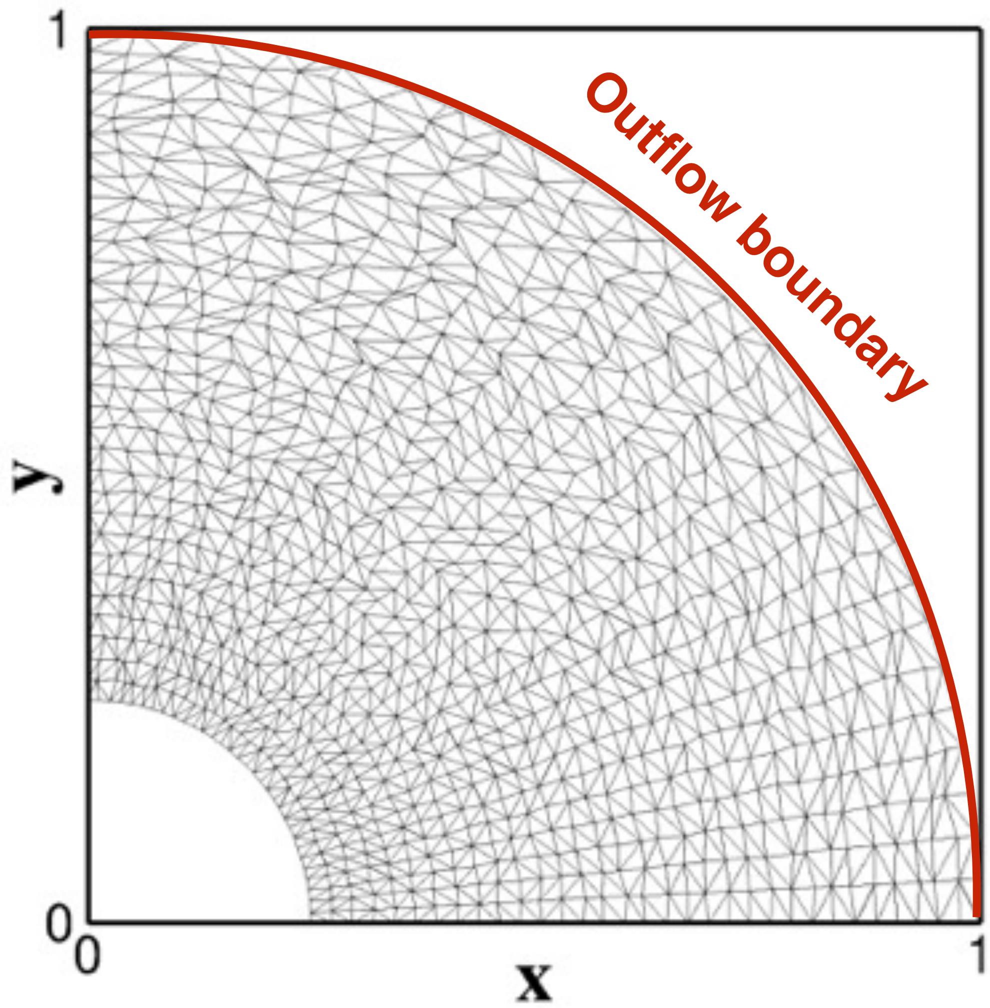


Irregular grids.

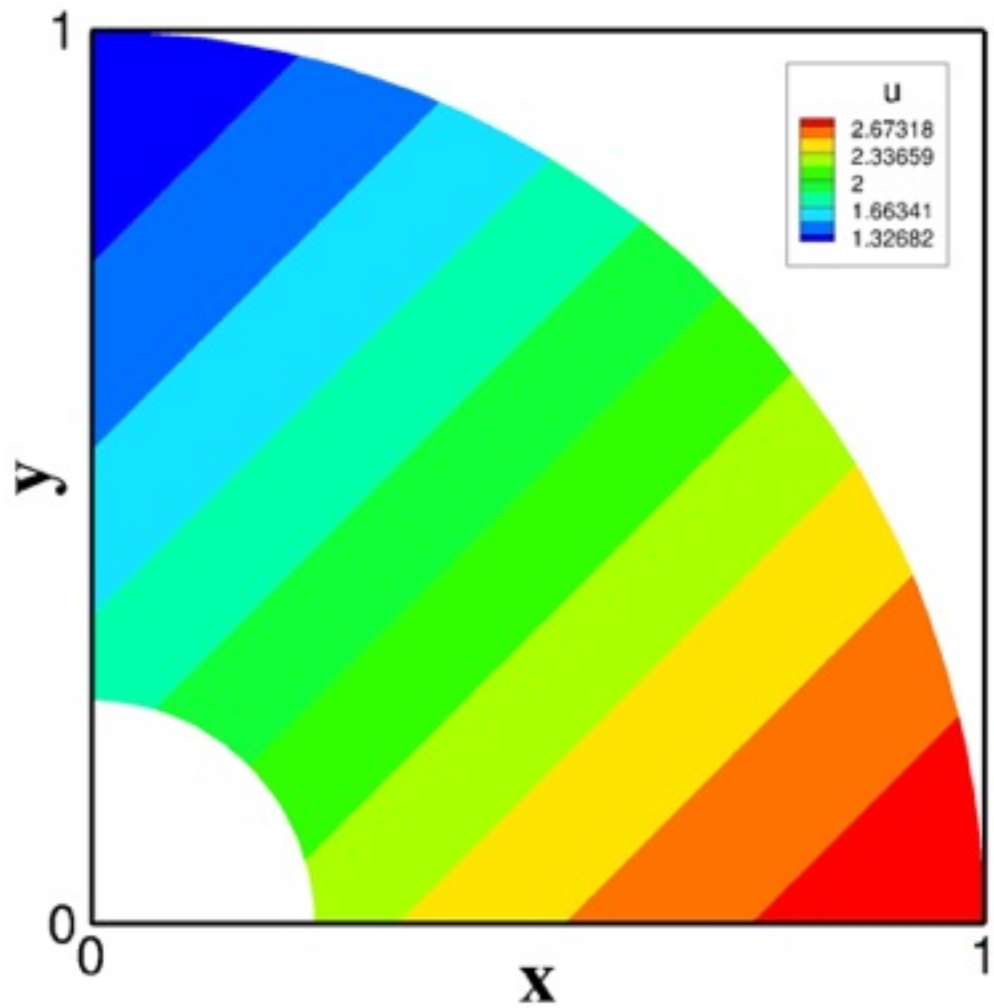
Compatibility condition is necessary to achieve 3rd-order DE.

2D Curved Domain

Burgers' equation, irregular grids in a curved domain:



(a) Irregular grid.

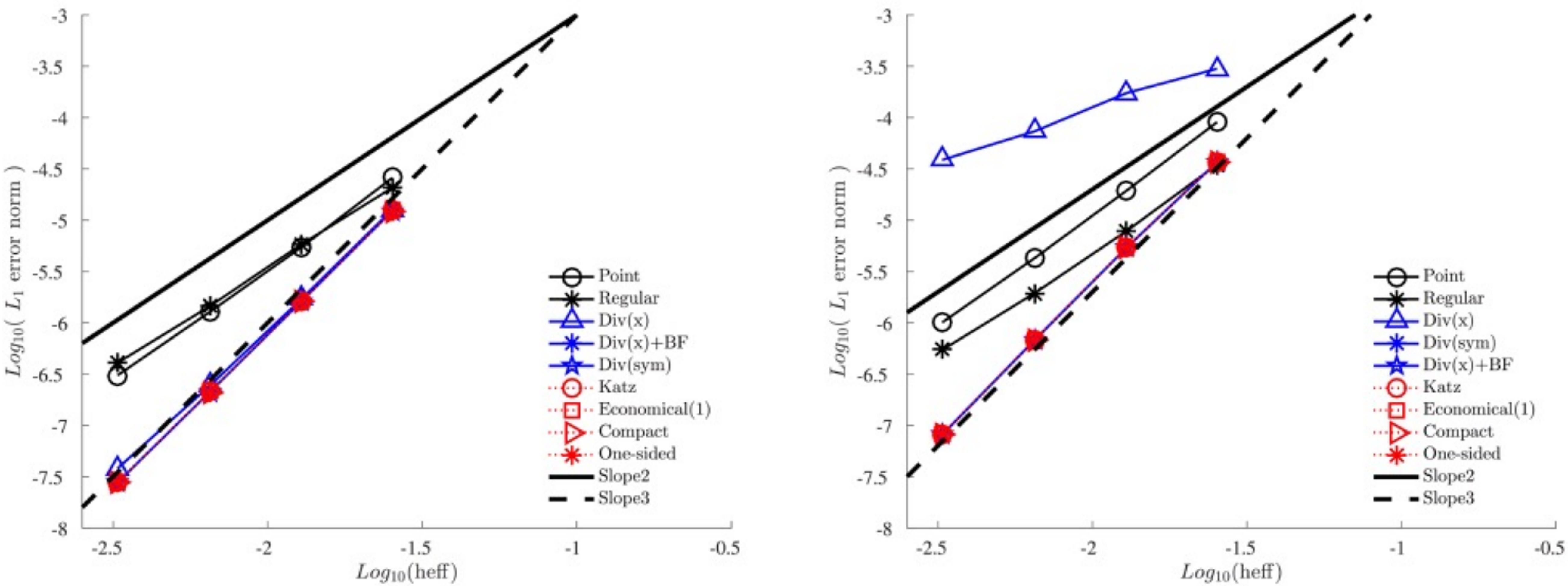


(b) Solution contours.

Irregular grids: nxn nodes, n = 33, 65, 129, 257

2D Curved Domain

Div(x) and Div(sym): no boundary closure
Div(x)-BF : use a boundary closure



(a) Interior nodes.

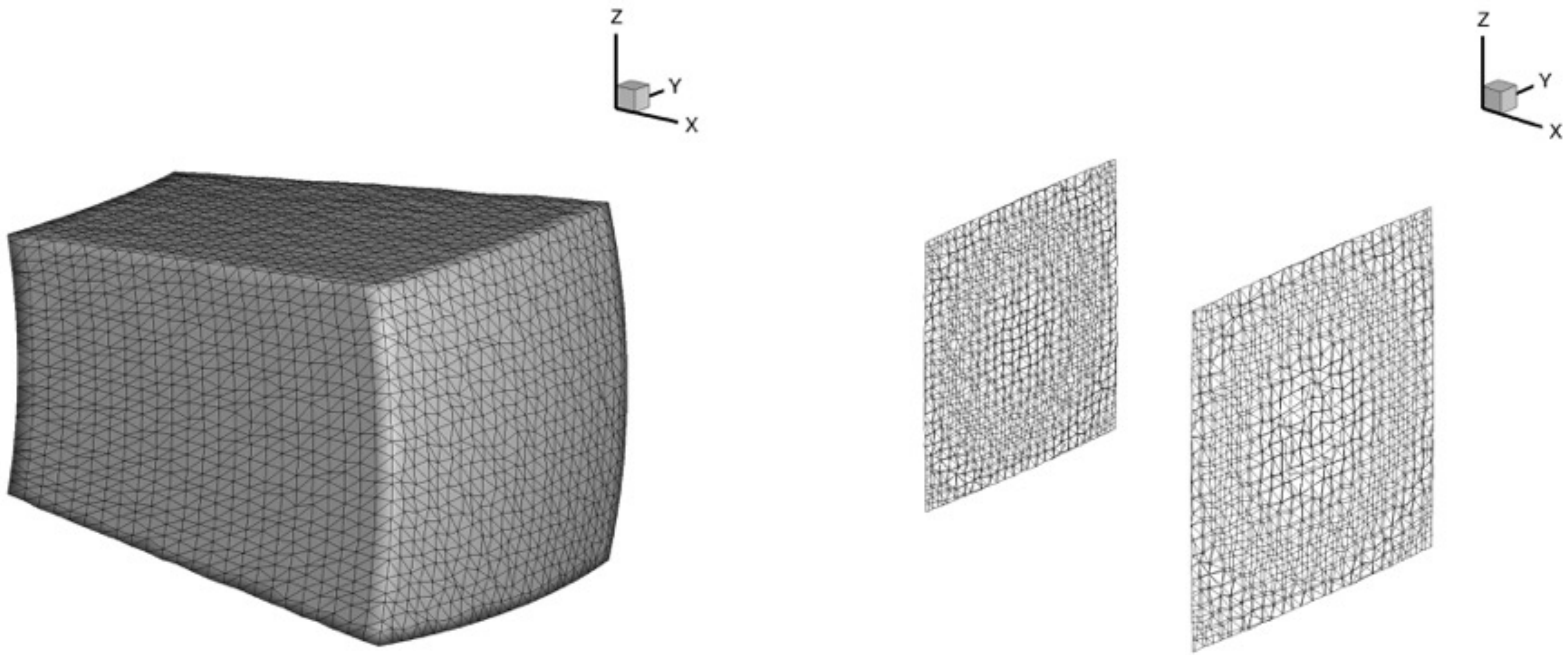
(b) Outflow boundary nodes.

Boundary closure is needed for Div(x), but not for all others.

3D Irregular Tetrahedral Grids

National Institute of Aerospace

Euler equations with manufactured solutions.



(a) Irregular tetrahedral grid (25×25×25).

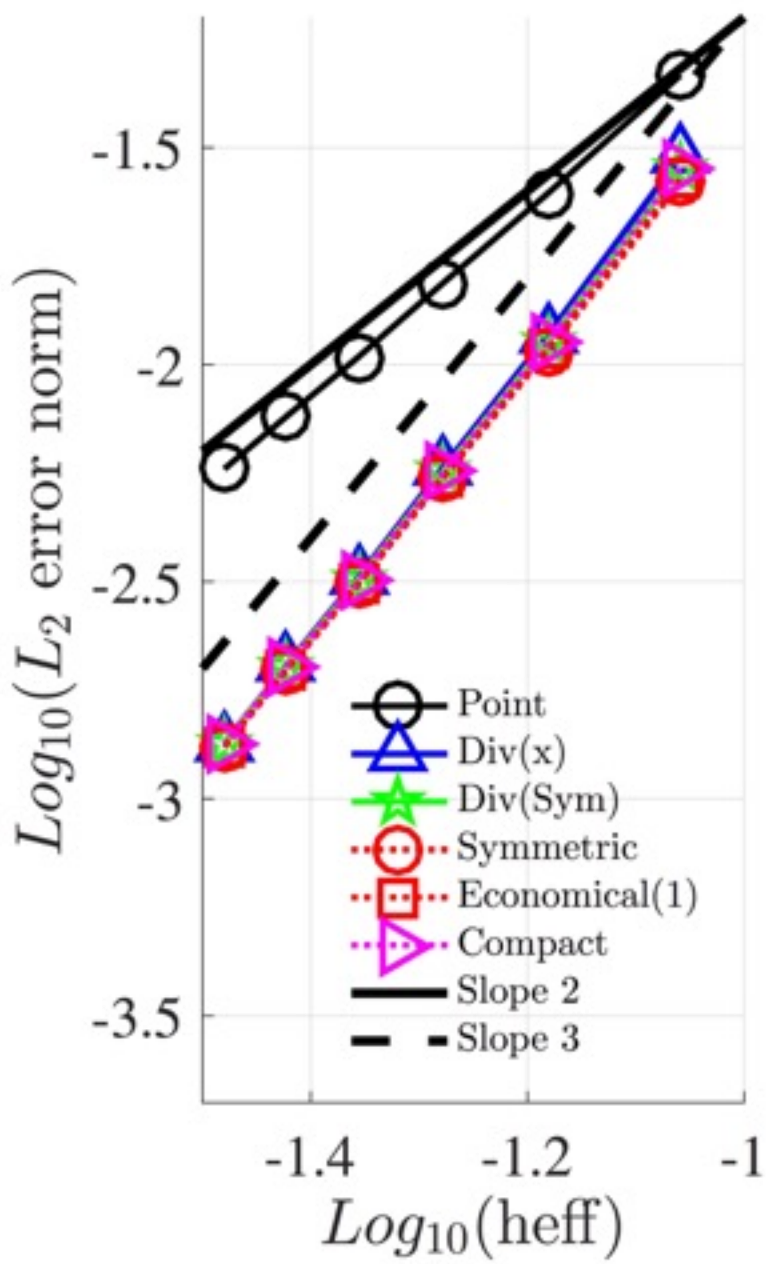
(b) Sections of the irregular tetrahedral grid.

Irregular grids: nxnxn nodes, n = 15, 20, 25, 30, 35 40.

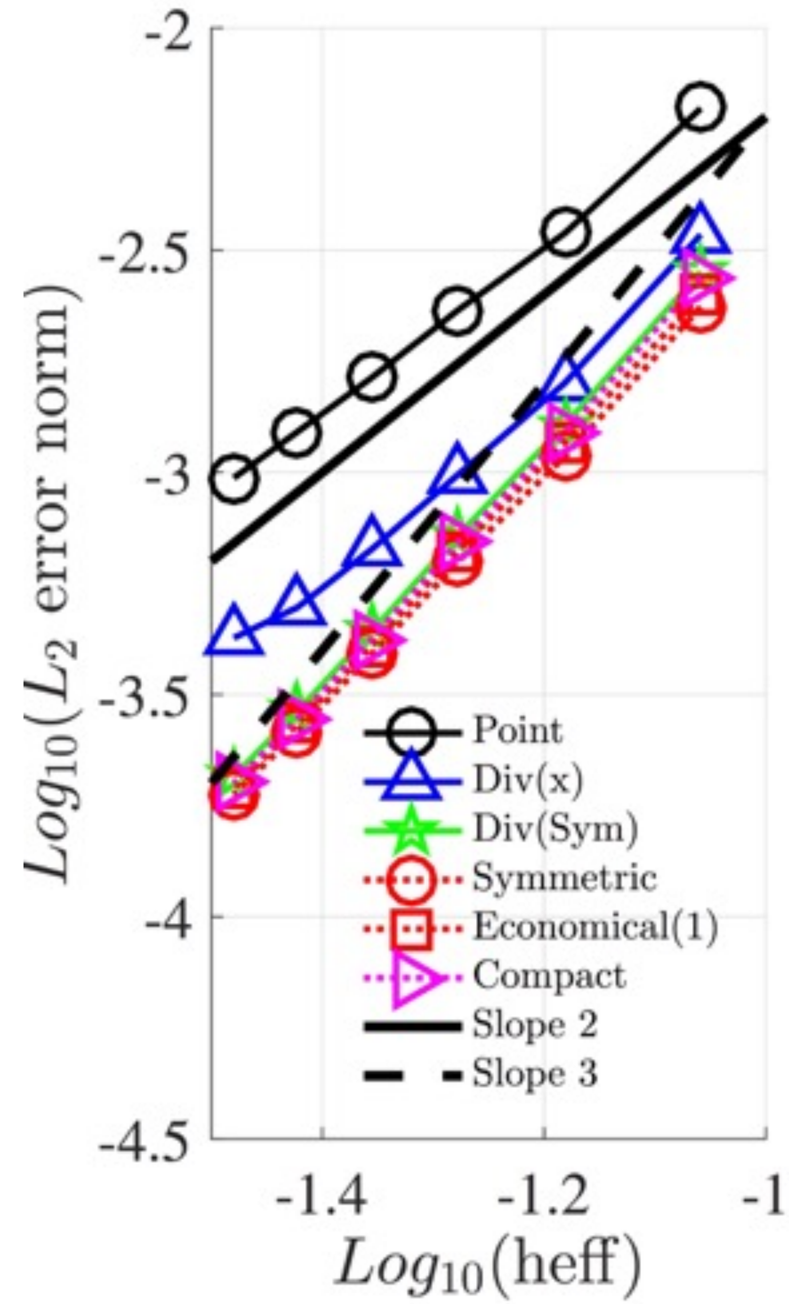
3D Irregular Tetrahedral Grids

NATIONAL INSTITUTE OF AEROSPACE

Error convergence for the x-velocity component.



Interior nodes



Boundary nodes

Div(x) needs boundary closure.

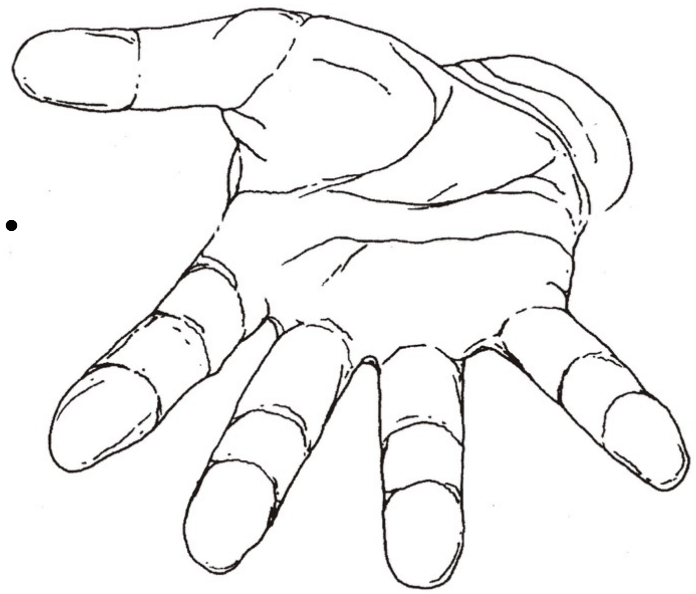
Conclusions

NATIONAL INSTITUTE OF AEROSPACE

- **Conditions identified for 3rd-order EB method.**

  (1)Vanishing 1st-order error.
  (2)Compatibility condition on regular grids.

  Follow the rules and pursue your self-interest.

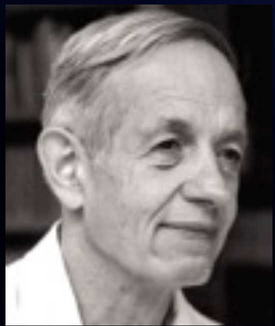


- **A family of source quadrature formulas derived.**

  Infinitely many accuracy-preserving formulas exist.

- **A unique and practical formula derived:** *Compact.*

A Great step towards low-cost 3rd-order unstructured-grid schemes for unsteady problems.



Nash Equilibrium or EB

NATIONAL INSTITUTE OF AEROSPACE

Dr. John Nash

“The Best for the Group comes when everyone in the group does what's best for himself AND the group.”   John Nash

In CFD…

EB (Low cost):
TE=O(h^2) that cancels

‘Best’ (High cost):
TE=O(h^3)

		Source/Viscous	
		EB(3rd)	‘Best’(3rd)
Inviscid	EB(3rd)	(3rd, 3rd) **Low cost**	(2nd, 2nd)
Inviscid	‘Best’(3rd)	(2nd, 2nd)	***Nash Equilibrium*** (3rd, 3rd) **High cost**

Accuracy (Inviscid, Source/Viscous)

A more economical 3rd-order scheme is achieved, when we cooperate and be compatible with each other.