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A brief introduction to summation-by-parts methods and their various flavors

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- Objective: To construct numerical methods for the solution of PDEs that are efficient, flexible, and robust
- Components of numerical methods for high-performance scientific computing that are efficient over their life cycle
  - Efficiency: high-order, optimized schemes, ...
  - Flexibility: Adaptivity, complex geometries, architecture aware implementations, ...
  - Robustness: Linear and nonlinear stability, ...
- Interest: Methods having the summation-by-parts property (SBP) regardless of how that property was achieved
- SBP+SATs provide a framework with the tools to construct the above-mentioned components

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- Consider the periodic advection equation:
$$\frac{\partial \mathcal{U}}{\partial t} + a\frac{\partial \mathcal{U}}{\partial x} = 0, \quad x \in [x_{\mathrm{L}}, x_{\mathrm{R}}],\; t \geq 0,\; \mathcal{U}(x_{\mathrm{R}}, t) = \mathcal{U}(x_{\mathrm{R}}, t),\; \mathcal{U}(x, 0) = \mathcal{F}(x)$$
- Is this problem (PDE+data) stable, i.e., solution is bounded by the data?
- Energy method: $\int\limits_{x_{\mathrm{L}}}^{x_{\mathrm{R}}} \mathcal{U}\left(\frac{\partial \mathcal{U}}{\partial t} + a\frac{\partial \mathcal{U}}{\partial x}\right)\mathrm{d}x = 0$
- Integration by parts: $\int\limits_{x_{\mathrm{L}}}^{x_{\mathrm{R}}} \mathcal{V}\frac{\partial \mathcal{U}}{\partial x} + \mathcal{U}\frac{\partial \mathcal{V}}{\partial \mathcal{U}}\mathrm{d}x = \oint_{\Gamma} \mathcal{V}\mathcal{U} n_x \mathrm{d}\Gamma = \mathcal{V}\mathcal{U}\Big|_{x_{\mathrm{L}}}^{x_{\mathrm{R}}}$

$$\therefore \int\limits_{x_{\mathrm{L}}}^{x_{\mathrm{R}}} \mathcal{U}\frac{\partial \mathcal{U}}{\partial \mathcal{U}} = \frac{1}{2}\mathcal{U}^2\Big|_{x_{\mathrm{L}}}^{x_{\mathrm{R}}} \text{ and } \int\limits_{x_{\mathrm{L}}}^{x_{\mathrm{R}}} \mathcal{U}\frac{\partial \mathcal{U}}{\partial t}\mathrm{d}t = \int\limits_{x_{\mathrm{L}}}^{x_{\mathrm{R}}} \frac{1}{2}\frac{\partial \mathcal{U}^2}{\partial t}\mathrm{d}x \xrightarrow{\text{Leibniz rule}} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{x_{\mathrm{L}}}^{x_{\mathrm{R}}} \mathcal{U}^2\mathrm{d}x$$

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- $$\frac{\mathrm{d}\|\mathcal{U}\|^2}{\mathrm{d}t} = -a\,\mathcal{U}^2\Big|_{x_\mathrm{L}}^{x_\mathrm{R}} \text{ where } \|\mathcal{U}\|^2 = \int_{x_\mathrm{L}}^{x_\mathrm{R}} \mathcal{U}^2 \mathrm{d}x$$

$$\|\mathcal{U}(\cdot,t)\|^2 - \|\mathcal{U}(\cdot,0)\|^2 = -a\int_0^t \mathcal{U}^2(x_\mathrm{R},\tau) - \mathcal{U}^2(x_\mathrm{L},\tau)\mathrm{d}\tau$$

$$\|\mathcal{U}(\cdot,t)\|^2 = \|\mathcal{F}(\cdot)\|^2 - a\int_0^t \mathcal{U}^2(x_\mathrm{R},\tau) - \mathcal{U}^2(x_\mathrm{L},\tau)\mathrm{d}\tau$$

$$\|\mathcal{U}(\cdot,t)\|^2 = \|\mathcal{F}(\cdot)\|^2$$

$\therefore$ stable

Energy method

- We can recast the convection equation in variational form as: find $\mathcal{U} \in \mathscr{V}$ such that for all $\mathcal{V} \in \mathscr{V}$ the following holds:

$$\int_{x_L}^{x_R} \mathcal{V}\frac{\partial \mathcal{U}}{\partial t} - \mathcal{U}\frac{\partial \mathcal{V}}{\partial x}\mathrm{d}x + \oint \mathcal{V}\mathcal{U}n_x \mathrm{d}\Gamma = 0$$

- We can perform the energy analysis starting from this form by replacing $\mathcal{V}$ with $\mathcal{U}$
- The energy method gives a simple means of proving the stability of a PDE + data or determining the correct data for a stable problem
- If we can mimic the energy method at the discrete level then we can apply the same type of analysis
- The key is integration by parts which allows the introduction of the boundary conditions (which in the variational form above has already been used to introduce the surface integral term)

Summation by parts

- The energy method can naturally be applied to finite-element methods
- The original intent of classical finite-difference (FD) SBP was to equip FD operators with integration by parts
- It turns out that constructing operators with this property allows for much more than provable linearly stability
  - Nonlinear stability proofs
  - Subcell finite volume interpretation
  - Dual-consistent discretizations
- The SBP framework gives a template to analyze new schemes so that they retain the desirable properties of the original schemes
  - Staggered formulations
  - Nonconforming blocks or elements
  - Adaptive methods, etc.

A matrix difference operator, $\mathsf{D}_x \in \mathbb{R}^{N\times N}$, is an SBP operator approximating the derivative $\frac{\partial}{\partial x}$, on the nodal distribution $\boldsymbol{x} \in [x_{\mathrm{L}}, x_{\mathrm{R}}]$, of order and degree p if

1. $\mathsf{D}_x \boldsymbol{x}^k = \mathsf{P}_x^{-1}\mathsf{Q}_x \boldsymbol{x}^k = \mathsf{P}_x^{-1}\left(\mathsf{S}_x + \frac{1}{2}\mathsf{E}_x\right)\boldsymbol{x}^k = k\boldsymbol{x}^{k-1},\ k = 0, 1, 2, \ldots, p;$
2. P_x, denoted the norm matrix, is symmetric positive definite;
3. $\mathsf{E}_x = \mathsf{E}_x^{\mathrm{T}}$, $\mathsf{S}_x = -\mathsf{S}_x^{\mathrm{T}}$, therefore, $\mathsf{Q}_x + \mathsf{Q}_x^{\mathrm{T}} = \mathsf{E}_x$; and
4. $\left(\boldsymbol{x}^m\right)^{\mathrm{T}} \mathsf{E}_x \boldsymbol{x}^k = \oint x^{m+k} n_x \mathrm{d}\Gamma = x^{m+k}\Big|_{x_{\mathrm{L}}}^{x_{\mathrm{R}}}$, $m, k = 0, 1, 2, \ldots, r$ $r \geq p$, where n_x is the x-component of the outward-facing unit normal.

- Here I use the following simple decomposition of E_x:

$$\mathsf{E}_x = \boldsymbol{t}_{\mathrm{R}}\boldsymbol{t}_{\mathrm{R}}^{\mathrm{T}} - \boldsymbol{t}_{\mathrm{L}}\boldsymbol{t}_{\mathrm{L}}^{\mathrm{T}},$$
$$\boldsymbol{t}_{\mathrm{R}}^{\mathrm{T}}\boldsymbol{x}^k = x_{\mathrm{R}}^k, \quad \boldsymbol{t}_{\mathrm{L}}^{\mathrm{T}}\boldsymbol{x}^k = x_{\mathrm{L}}^k, \quad k = 0, 1, 2, \ldots, r$$

- The above definition does not necessitate nodes on the boundaries of the domain or nodal distributions contained strictly within the boundaries of the element

- One can show that:

$$\boldsymbol{v}^{\mathrm{T}}\mathsf{P}_x\boldsymbol{u} \approx \int_{x_{\mathrm{L}}}^{x_{\mathrm{R}}} \mathcal{V}\mathcal{U}\mathrm{d}x, \quad \boldsymbol{v}^{\mathrm{T}}\mathsf{Q}_x\boldsymbol{u} \approx \int_{x_{\mathrm{L}}}^{x_{\mathrm{R}}} \mathcal{V}\frac{\partial \mathcal{U}}{\partial x}\mathrm{d}x, \quad \boldsymbol{v}^{\mathrm{T}}\mathsf{Q}_x^{\mathrm{T}}\boldsymbol{u} \approx \int_{x_{\mathrm{L}}}^{x_{\mathrm{R}}} \mathcal{U}\frac{\partial \mathcal{V}}{\partial x}\mathrm{d}x,$$

$$\boldsymbol{v}^{\mathrm{T}}\mathsf{S}_x\boldsymbol{u} \approx \int_{x_{\mathrm{L}}}^{x_{\mathrm{R}}} \mathcal{V}\frac{\partial \mathcal{U}}{\partial x}\mathrm{d}x - \frac{1}{2}\oint \mathcal{V}\mathcal{U}n_x\mathrm{d}\Gamma, \quad \boldsymbol{v}^{\mathrm{T}}\mathsf{S}_x^{\mathrm{T}}\boldsymbol{u} \approx \int_{x_{\mathrm{L}}}^{x_{\mathrm{R}}} \mathcal{U}\frac{\partial \mathcal{V}}{\partial x}\mathrm{d}x - \frac{1}{2}\oint \mathcal{V}\mathcal{U}n_x\mathrm{d}\Gamma$$

- How does this lead to a high-order approximation to integration by parts?

Integration by parts: $\int_{x_{\mathrm{L}}}^{x_{\mathrm{R}}} \mathcal{V}\frac{\partial \mathcal{U}}{\partial x} + \mathcal{U}\frac{\partial \mathcal{V}}{\partial \mathcal{U}}\mathrm{d}x = \oint \mathcal{V}\mathcal{U}n_x\mathrm{d}\Gamma = \mathcal{V}\mathcal{U}|_{x_{\mathrm{L}}}^{x_{\mathrm{R}}}$

Summation by parts: $\boldsymbol{v}^{\mathrm{T}}\mathsf{P}_x\mathsf{D}_x\boldsymbol{u} + \boldsymbol{u}^{\mathrm{T}}\mathsf{P}_x\mathsf{D}_x\boldsymbol{v} = \boldsymbol{v}^{\mathrm{T}}\mathsf{E}_x\boldsymbol{u}$

Variational and strong form discretizations

- To discretize we decompose the domain Ω into L elements: $\Omega = \bigcup_{l=1}^{L} \Omega_l$

$$\text{Strong form: } \frac{\partial \mathcal{U}_l}{\partial t} + a\frac{\partial \mathcal{U}_l}{\partial x} = 0, \quad x \in [x_l, x_{l+1}], t \geq 0$$

$$\text{Variational form: } \int_{x_l}^{x_{l+1}} \mathcal{V}_l \frac{\partial \mathcal{U}_l}{\partial t} - a\mathcal{U}_l \frac{\partial \mathcal{V}_l}{\partial x} \mathrm{d}x + a \oint \mathcal{V}_l \mathcal{U}_l n_x \mathrm{d}\Gamma = 0$$

- To insert coupling between adjacent elements in the surface integral we replace $\mathcal{U}_l$ with a numerical flux $\mathcal{U}_l^*$
- The SBP discretization reads, find $\boldsymbol{u}_l \in \mathbb{R}^N$ such that for all $\boldsymbol{v}_l \in \mathbb{R}^N$ the following holds:

$$\boldsymbol{v}_l^{\mathrm{T}} \mathsf{P}_x \frac{\mathrm{d}\boldsymbol{u}_l}{\mathrm{d}t} - a\boldsymbol{v}_l^{\mathrm{T}} \mathsf{Q}_x^{\mathrm{T}} \boldsymbol{u}_l + a\boldsymbol{v}_l^{\mathrm{T}} \boldsymbol{t}_{\mathrm{R}} \boldsymbol{u}_{l+1}^* - a\boldsymbol{v}_l^{\mathrm{T}} \boldsymbol{t}_{\mathrm{L}} \boldsymbol{u}_l^* = 0$$

- Two common numerical flux functions are:

$$\text{Central flux: } a\boldsymbol{u}_l^* \equiv \frac{a}{2}\left(\boldsymbol{t}_{\mathrm{L}}^{\mathrm{T}} \boldsymbol{u}_l + \boldsymbol{t}_{\mathrm{R}}^{\mathrm{T}} \boldsymbol{u}_{l-1}\right)$$

$$\text{Upwind flux: } a\boldsymbol{u}_l^* \equiv \frac{a}{2}\left(\boldsymbol{t}_{\mathrm{L}}^{\mathrm{T}} \boldsymbol{u}_l + \boldsymbol{t}_{\mathrm{R}}^{\mathrm{T}} \boldsymbol{u}_{l-1}\right) - \frac{|a|}{2}\left(\boldsymbol{t}_{\mathrm{L}}^{\mathrm{T}} \boldsymbol{u}_l - \boldsymbol{t}_{\mathrm{R}}^{\mathrm{T}} \boldsymbol{u}_{l-1}\right)$$

- An instructive means of obtaining the strong-form discretization is to apply integration by parts a second time to the variatoinal form (this is the same approach as in the nodal DG of Hesthaven and Warburton) which gives

$$\int_{x_{\mathrm{L}}}^{x_{\mathrm{R}}} \mathcal{V}_l \frac{\partial \mathcal{U}_l}{\partial t} + a\mathcal{V}_l \frac{\partial \mathcal{U}_l}{\partial x} \mathrm{d}x + a \oint \mathcal{V}_l (\mathcal{U}_l^* - \mathcal{U}_l) \mathrm{d}\Gamma = 0,$$

the semi-discrete equations are

$$\boldsymbol{v}_l^{\mathrm{T}} \mathsf{P}_x \frac{\mathrm{d}\boldsymbol{u}_l}{\mathrm{d}t} + a\boldsymbol{v}_l^{\mathrm{T}} \mathsf{Q}_x \boldsymbol{u}_l + a\boldsymbol{v}_l^{\mathrm{T}} \boldsymbol{t}_{\mathrm{R}} (\boldsymbol{u}_{l+1}^* - \boldsymbol{t}_{\mathrm{R}}^{\mathrm{T}} \boldsymbol{u}_l) - a\boldsymbol{v}_l^{\mathrm{T}} \boldsymbol{t}_{\mathrm{L}} (\boldsymbol{u}_l^* - \boldsymbol{t}_{\mathrm{R}}^{\mathrm{T}} \boldsymbol{u}_l) = 0$$

since this must hold for all $\boldsymbol{v}_l \in \mathbb{R}^N$

$$\frac{\mathrm{d}\boldsymbol{u}_l}{\mathrm{d}t} + a\mathsf{D}_x \boldsymbol{u}_l + a\mathsf{P}_x^{-1} \boldsymbol{t}_{\mathrm{R}} (\boldsymbol{u}_{l+1}^* - \boldsymbol{t}_{\mathrm{R}}^{\mathrm{T}} \boldsymbol{u}_l) - a\mathsf{P}_x^{-1} \boldsymbol{t}_{\mathrm{L}} (\boldsymbol{u}_l^* - \boldsymbol{t}_{\mathrm{R}}^{\mathrm{T}} \boldsymbol{u}_l) = 0,$$

- The variational and strong form are equivalent for linear problems

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- I will show stability using the variational form for the upwind numerical flux
- Set $\boldsymbol{v}_l = \boldsymbol{u}_l$

$$\boldsymbol{u}_l^{\mathrm{T}} \mathrm{P}_x \frac{\mathrm{d}\boldsymbol{u}_l}{\mathrm{d}t} - a \boldsymbol{u}_l^{\mathrm{T}} \mathrm{Q}_x^{\mathrm{T}} + \boldsymbol{u}_l^{\mathrm{T}} \boldsymbol{t}_{\mathrm{R}} \left[\frac{a}{2} \left(\boldsymbol{t}_{\mathrm{L}}^{\mathrm{T}} \boldsymbol{u}_{l+1} + \boldsymbol{t}_{\mathrm{R}}^{\mathrm{T}} \boldsymbol{u}_l \right) - \frac{|a|}{2} \left(\boldsymbol{t}_{\mathrm{L}}^{\mathrm{T}} \boldsymbol{u}_{l+1} - \boldsymbol{t}_{\mathrm{R}}^{\mathrm{T}} \boldsymbol{u}_l \right) \right]$$

$$\boldsymbol{u}_l^{\mathrm{T}} \boldsymbol{t}_{\mathrm{L}} \left[\frac{a}{2} \left(\boldsymbol{t}_{\mathrm{L}}^{\mathrm{T}} \boldsymbol{u}_l + \boldsymbol{t}_{\mathrm{R}}^{\mathrm{T}} \boldsymbol{u}_{l-1} \right) - \frac{|a|}{2} \left(\boldsymbol{t}_{\mathrm{L}}^{\mathrm{T}} \boldsymbol{u}_l - \boldsymbol{t}_{\mathrm{R}}^{\mathrm{T}} \boldsymbol{u}_{l-1} \right) \right] = 0$$

$$\mathrm{Q}_x^{\mathrm{T}} = \mathrm{S}_x^{\mathrm{T}} + \frac{1}{2} \mathrm{E}_x \quad \boldsymbol{u}_l^{\mathrm{T}} \mathrm{S}_x^{\mathrm{T}} \boldsymbol{u}_l = 0$$

$$\frac{1}{2} \frac{\mathrm{d} \|\boldsymbol{u}_l\|_{\mathrm{P}_x}^2}{\mathrm{d}t} = -\frac{(a - |a|)}{2} ul^{\mathrm{T}} \boldsymbol{t}_{\mathrm{R}} \boldsymbol{t}_{\mathrm{L}}^{\mathrm{T}} \boldsymbol{u}_{l+1} + \frac{(a + |a|)}{2} \boldsymbol{u}_l^{\mathrm{T}} \boldsymbol{t}_{\mathrm{L}} \boldsymbol{t}_{\mathrm{R}}^{\mathrm{T}} \boldsymbol{u}_{l-1} - \frac{|a|}{2} \left(\boldsymbol{u}_l^{\mathrm{T}} \boldsymbol{t}_{\mathrm{R}} \boldsymbol{t}_{\mathrm{R}}^{\mathrm{T}} \boldsymbol{u}_l + \boldsymbol{u}_l^{\mathrm{T}} \boldsymbol{t}_{\mathrm{L}} \boldsymbol{t}_{\mathrm{L}}^{\mathrm{T}} \boldsymbol{u}_l \right)$$

where $\|\boldsymbol{u}_l\|_{\mathrm{P}_x}^2 = \boldsymbol{u}_l^{\mathrm{T}} \mathrm{P}_x \boldsymbol{u}_l$

- Summing over the elements and after rearrangement, cancellation, etc., we obtain

$$\sum_{l=1}^{L} \frac{\mathrm{d}\|\boldsymbol{u}_l\|_{\mathrm{P}_x}^2}{\mathrm{d}t} = |a| \left[\sum_{l=1}^{L-1} \tilde{\boldsymbol{u}}_l^{\mathrm{T}} \mathrm{M} \tilde{\boldsymbol{u}}_l + \tilde{\boldsymbol{u}}_L^{\mathrm{T}} \mathrm{M} \tilde{\boldsymbol{u}}_L \right]$$

where $\tilde{\boldsymbol{u}}_l^{\mathrm{T}} = \left[\boldsymbol{u}_l^{\mathrm{T}} \boldsymbol{t}_{\mathrm{R}}, \boldsymbol{u}_{l+1}^{\mathrm{T}} \boldsymbol{t}_{\mathrm{L}} \right]$, $\tilde{\boldsymbol{u}}_L^{\mathrm{T}} = \left[\boldsymbol{u}_L^{\mathrm{T}} \boldsymbol{t}_{\mathrm{R}}, \boldsymbol{u}_1^{\mathrm{T}} \boldsymbol{t}_{\mathrm{L}} \right]$ and $\mathrm{M} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

- Since M is negative semidefinite

$$\sum_{l=1}^{L} \frac{\mathrm{d}\|\boldsymbol{u}_l\|_{\mathrm{P}_x}^2}{\mathrm{d}t} \leq 0,$$

integrating in time and applying the initial condition,

$$\sum_{l=1}^{L} \|\boldsymbol{u}_l\|_{\mathrm{P}_x}^2 \leq \sum_{l=1}^{L-1} \|\boldsymbol{f}_l\|_{\mathrm{P}_x}^2$$

Conservation

- Element-wise conservation can be proven by setting $\boldsymbol{v}_l = \mathbf{1}$, which results in

$$\sum_{l=1}^{L} \mathbf{1}^{\mathrm{T}} \mathsf{P}_x \frac{\mathrm{d}\boldsymbol{u}_l}{\mathrm{d}t} = 0$$

- Alternatively, Fisher et al. (2013) have shown that tensor-product SBP operators can be recast in flux-differencing form and conservation, in the sense of Lax-Wendroff, can be easily proven from this form

Flavor: Traditional finite difference

- Applied to nodal distributions with essentially arbitrary numbers of nodes while retaining their point-wise error characteristics
- The domain is broken into a number of blocks of fixed size and mesh refinement is accomplished by increasing the number of nodes within blocks
- Simple example on N nodes

$$\mathrm{D}_x = \frac{N-1}{x_{\mathrm{R}} - x_{\mathrm{L}}} \begin{bmatrix} -1 & 1 & & & \\ -\frac{1}{2} & 0 & \frac{1}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{1}{2} & 0 & \frac{1}{2} \\ & & & -1 & 1 \end{bmatrix}, \mathrm{Q}_x = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & & & \\ -\frac{1}{2} & 0 & \frac{1}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{1}{2} & 0 & \frac{1}{2} \\ & & & -\frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

$$\mathrm{P}_x = \frac{x_{\mathrm{R}} - x_{\mathrm{L}}}{N-1} \begin{bmatrix} \frac{1}{2} & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \frac{1}{2} \end{bmatrix}$$

Flavor: Discontinuous element

- The domain is broken up into a number of elements each having a fixed number of nodes
- Mesh refinement is accomplished by increasing the number of elements
- A simple example: Legendre-Gauss-Lobatto $\boldsymbol{\xi} = \left[-1, -1\frac{\sqrt{5}}{5}, \frac{\sqrt{5}}{5}, 1\right]^{\mathrm{T}}$

$$
\mathsf{D}_x = \frac{2}{x_{\mathrm{R}} - x_{\mathrm{L}}} \begin{bmatrix} -3 & \frac{5}{4} + \frac{5}{4}\sqrt{5} & \frac{5}{4} - \frac{5}{4}\sqrt{5} & \frac{1}{2} \\ -\frac{1}{4} - \frac{1}{4}\sqrt{5} & 0 & \frac{1}{2}\sqrt{5} & \frac{1}{4} - \frac{1}{4}\sqrt{5} \\ \frac{1}{4}\sqrt{5} - \frac{1}{4} & -\frac{1}{2}\sqrt{5} & 0 & \frac{1}{4}\sqrt{5} + \frac{1}{4} \\ -\frac{1}{2} & -\frac{5}{4} + \frac{5}{4}\sqrt{5} & -\frac{5}{4} - \frac{5}{4}\sqrt{5} & 3 \end{bmatrix},
$$

$$
\mathsf{Q}_x = \begin{bmatrix} -\frac{1}{2} & \frac{5}{24} + \frac{5}{24}\sqrt{5} & \frac{5}{24} - \frac{5}{24}\sqrt{5} & \frac{1}{12} \\ -\frac{5}{24} - \frac{5}{24}\sqrt{5} & 0 & \frac{5}{12}\sqrt{5} & \frac{5}{24} - \frac{5}{24}\sqrt{5} \\ -\frac{5}{24} + \frac{5}{24}\sqrt{5} & -\frac{5}{12}\sqrt{5} & 0 & \frac{5}{24} - \frac{5}{24}\sqrt{5} \\ -\frac{1}{12} & -\frac{5}{24} + \frac{5}{24}\sqrt{5} & -\frac{5}{24} - \frac{5}{24}\sqrt{5} & \frac{1}{2} \end{bmatrix},
$$

$$
\mathsf{P}_x = \frac{x_{\mathrm{R}} - x_{\mathrm{L}}}{2} \mathrm{diag}\left(\frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{1}{6}\right)
$$

- Approximate the solution using the Lagrangian interpolant: $\mathcal{U}_l \equiv \sum_{k=0}^{N-1} \mathcal{L}_k(\xi)\mathcal{U}_l(\xi_k)$
- Semi-discrete equations

$$\int_{-1}^{1} \mathcal{J}\mathcal{L}_i \frac{\partial \mathcal{U}_l}{\partial t} - a\mathcal{U}_l \frac{\mathrm{d}\mathcal{L}_i}{\mathrm{d}\xi}\mathrm{d}\xi + a\mathcal{L}_i(1)\mathcal{U}_h^*(1) - a\mathcal{L}_i(-1)\mathcal{U}_l^*(-1) = 0, \quad i = 0, 1, 2, \ldots, N-1$$

$$\mathcal{J} = \frac{(x_\mathrm{R} - x_\mathrm{L})}{2}$$

- In matrix notation

$$\mathcal{J}\mathrm{P}_\xi \frac{\mathrm{d}\boldsymbol{u}_l}{\mathrm{d}t} - a\mathrm{Q}_\xi^\mathrm{T}\boldsymbol{u}_l + a\boldsymbol{t}_\mathrm{R}\boldsymbol{u}_{l+1} - a\boldsymbol{t}_\mathrm{L}\boldsymbol{u}_{l-1} = 0$$

- where

$$(\mathrm{P}_x)_{ij} = \int_{-1}^{1} \mathcal{L}_i\mathcal{L}_j\mathrm{d}\xi, \quad (\mathrm{Q}_\xi)_{ij} = \int_{-1}^{1} \mathcal{L}_i \frac{\mathrm{d}\mathcal{L}_j}{\mathrm{d}\xi}\mathrm{d}\xi,$$

$$\boldsymbol{t}_\mathrm{R}^\mathrm{T} = [\mathcal{L}_0(1), \ldots, \mathcal{L}_{N-1}(1)], \quad \boldsymbol{t}_\mathrm{L}^\mathrm{T} = [\mathcal{L}_0(-1), \ldots, \mathcal{L}_{N-1}(-1)]$$

Flavor: Continuous element

- For nodal distributions that contain boundary nodes, we can collapse the operators and remove the double solution point at the element boundaries
- The global norm matrix, P_x, and stiffness matrix, Q_x, are constructed from the local matrices as

$$\mathsf{P}_x = \sum_{l=1}^{L} \mathsf{T}_l \left(\mathsf{P}_x\right)_i \mathsf{T}_l^{\mathrm{T}},$$

$$\mathsf{Q}_x = \sum_{l=1}^{L} \mathsf{T}_l \left(\mathsf{Q}_x\right)_i \mathsf{T}_l^{\mathrm{T}},$$

where T_l is of size $[L(N-1)+1] \times N$

$\mathsf{T}_l((l-1)(N-1)+1 : l*(N-1)+1, :) = \mathrm{diag}(1,\ldots,1)$

- Advantage: reduces the number of DOFs
- Disadvantage: underconvergence for even-degree (odd number of nodes) operators

Flavor: Hybrid element

- Combine the discontinuous element and continuous element approach
- Construct new elements that are a merger of discontinuous elements by using the continuous element approach
- This reduces the number of DOFs and cures the underconvergence of even-degree operators

Other flavors

- Staggered-grid formulations: Flux points and solutions points on different meshes
- Nonconforming elements:
- Upwind SBP operators:
- Flux reconstruction: Prove stability in a norm other than the natural norm; requires careful construction of the interelement coupling to preserve stability and conservation
- ...

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- Consider an open and bounded domain $\Omega \subset \mathbb{R}^d$ with a piecewise-smooth boundary Γ composed of n_Γ smooth subsurfaces. The matrix D_x is a degree p SBP approximation to the first derivative $\frac{\partial}{\partial x}$ on the nodes $S_\Omega \{\boldsymbol{x}_i\}_{i=1}^N$ if
  1. $\mathsf{D}_x \boldsymbol{p}_k = \left.\frac{\partial \mathcal{P}_k}{\partial x}\right|_{S_\Omega}, \ \forall k \in \left\{1, 2, \ldots, \binom{p+d}{d}\right\}$, where $\mathcal{P}_k$ is the k^{th} basis polynomial;
  2. $\mathsf{D}_x = \mathsf{P}^{-1}\mathsf{Q}_x$, where P is symmetric positive-definite;
  3. $\mathsf{Q}_x = \mathsf{S}_x + \frac{1}{2}\mathsf{E}_x$, where $\mathsf{S}_x = -\mathsf{S}_x^{\mathrm{T}}$, $\mathsf{E}_x = \mathsf{E}_x^{\mathrm{T}}$; and
  4. E_x satisfies

$$\boldsymbol{p}_k^{\mathrm{T}} \mathsf{E}_x \boldsymbol{p}_m = \oint_\Gamma \mathcal{P}_k \mathcal{P}_m n_x \mathrm{d}\Gamma, \quad \forall\, k, m \in \left\{1, 2, \ldots, \binom{r+d}{d}\right\}$$

  where $r \geq p$, and n_x is the x component of the outward pointing unit normal

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$$\text{Integration by parts: } \int_\Omega \mathcal{V}\frac{\partial \mathcal{U}}{\partial x} + \mathcal{U}\frac{\partial \mathcal{V}}{\partial x} \mathrm{d}\Omega = \oint_\Gamma \mathcal{V}\mathcal{U} n_x \mathrm{d}\Gamma$$

$$\text{Summation by parts: } \boldsymbol{v}^{\mathrm{T}}\mathsf{P}\mathsf{D}_x\boldsymbol{u} + \boldsymbol{v}^{\mathrm{T}}\mathsf{P}\mathsf{D}_x\boldsymbol{u} = \boldsymbol{v}^{\mathrm{T}}\mathsf{E}_x\boldsymbol{u}$$

- There is no restriction on the type of nodal distribution that you can consider

- Linear stability is not sufficient for the nonlinear problems we are interested in solving
- Instead we look to prove nonlinear stability; one approach is to look at secondary conservation laws, in particular the conservation of entropy
- We consider the generic conservation law

$$\frac{\partial \boldsymbol{\mathcal{Q}}}{\partial t} + \sum_{i=1}^{3} \frac{\partial \boldsymbol{\mathcal{F}}_i}{\partial x_i} = 0$$

Nonlinear stability: the mathematical entropy

- Then the convex scalar function $\mathcal{S}(\boldsymbol{\mathcal{Q}})$ is an entropy function of the above system if
  - Differentiating $\mathcal{S}$ w.r.t $\boldsymbol{\mathcal{Q}}$ simultaneously contracts all spatial fluxes as follows:

$$\frac{\partial \mathcal{S}}{\partial \boldsymbol{\mathcal{Q}}} \frac{\partial \boldsymbol{\mathcal{F}}_{\boldsymbol{i}}}{\partial x_i} = \boldsymbol{\mathcal{W}}^{\mathrm{T}} \frac{\partial \boldsymbol{\mathcal{F}}_{\boldsymbol{i}}}{\partial x_i} = \frac{\partial \mathcal{F}}{\partial x_i}, \quad i = 1, 2, 3$$

    where $\boldsymbol{\mathcal{W}}$ are the entropy variables and $\mathcal{F}_i$ are the entropy fluxes

  - The entropy variables symmetrize the system:

$$\frac{\partial \boldsymbol{\mathcal{Q}}}{\partial t} + \sum_{i=1}^{3} \frac{\partial \boldsymbol{\mathcal{F}}_i}{\partial x_i} = \frac{\partial \boldsymbol{\mathcal{Q}}}{\partial \boldsymbol{\mathcal{W}}} \frac{\partial \boldsymbol{\mathcal{W}}}{\partial t} + \sum_{i=1}^{3} \frac{\partial \boldsymbol{\mathcal{F}}_i}{\partial \boldsymbol{\mathcal{W}}} \frac{\partial \boldsymbol{\mathcal{W}}}{\partial x_i}$$

    with the symmetry conditions: $\frac{\partial \boldsymbol{\mathcal{Q}}}{\partial \boldsymbol{\mathcal{W}}} = \left(\frac{\partial \boldsymbol{\mathcal{Q}}}{\partial \boldsymbol{\mathcal{W}}}\right)^{\mathrm{T}}, \frac{\partial \boldsymbol{\mathcal{F}}_i}{\partial \boldsymbol{\mathcal{W}}} = \left(\frac{\partial \boldsymbol{\mathcal{F}}_i}{\partial \boldsymbol{\mathcal{W}}}\right)^{\mathrm{T}}$

- If $\mathcal{S}$ exists then we can construct a new conservation law on the entropy by contracting the conservation law with the entropy variables:

$$\boldsymbol{\mathcal{W}}^{\mathrm{T}}\left(\frac{\partial \boldsymbol{\mathcal{Q}}}{\partial t}+\sum_{i=1}^{3}\frac{\partial \boldsymbol{\mathcal{F}}_i}{\partial x_i}\right)=\frac{\partial \mathcal{S}}{\partial t}+\sum_{i=1}^{3}\frac{\partial \mathcal{F}_i}{\partial x_i}=0$$

- Integrating over the domain and using Leibniz rule:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\mathcal{S}\mathrm{d}\Omega=-\oint_{\Gamma}\sum_{i=1}^{3}\mathcal{F}_i n_{x_i}\mathrm{d}\Gamma$$

- The above can then be converted into a bound on the solution $\boldsymbol{\mathcal{Q}}$ (Dafermos 2000, Svärd 2015)
- As with linear stability, semi-discrete stability is proven by mimicking the continuous stability proof step by step
- Nonlinear difference operators can be constructed from SBP operators that result in discretely entropy conservative/stable schemes
- The resulting proofs do not rely on integral exactness

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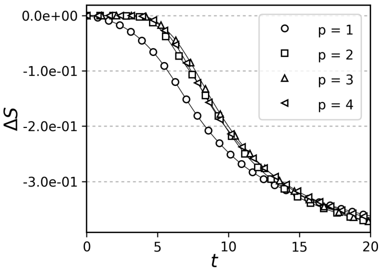
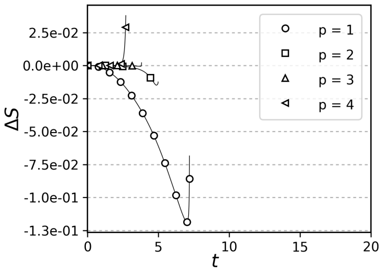
- Principle author, Jared Crean, Jason Hicken's stellar PhD student; will be presenting the details at Aviation
- Entropy stable discretization of the compressible Euler equations using multi-dimensional SBP operators on triangles (paper to come shortly)
- Taylor Green Vortex problem on a periodic cube $[-\pi, \pi]^3$



(a) Entropy stable algorithm



(b) Standard algorithm (Roe solver)

- The SBP concept provides a rigorous mathematical framework for discretizations of PDEs
- This framework provides guidance for the discretization of new problems
- To date much flexibility has been introduced
- The sky is the limit

Questions