Hyperbolic Navier-Stokes Method Based on Reconstructed-Discontinuous-Galerkin or Reconstructed-Finite-Volume Formulation with Variational Reconstruction

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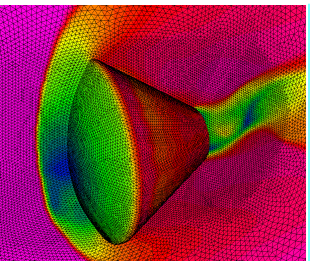
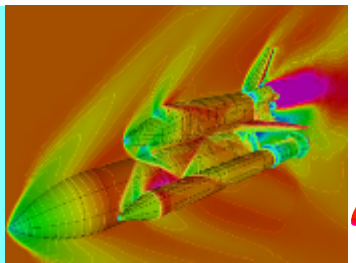
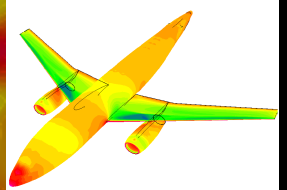
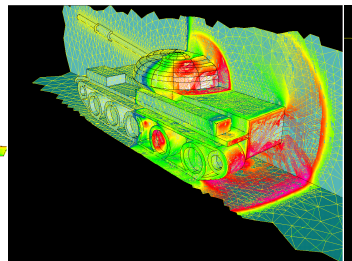
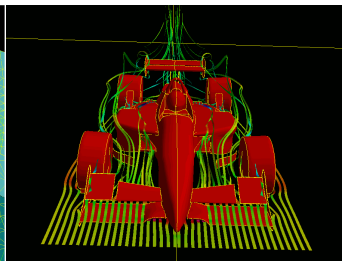


Background

Navier-Stokes Equations

Inviscid (Hyperbolic)

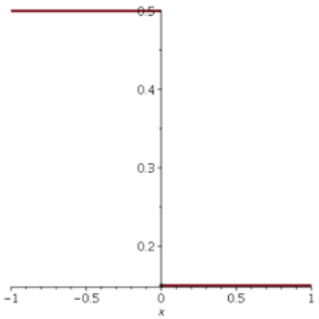
Viscous (Parabolic)

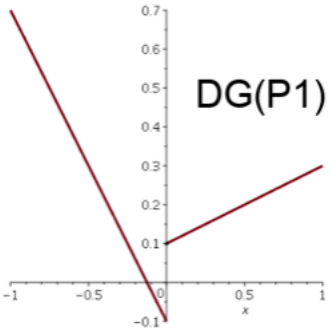
Finite Volume (FV)

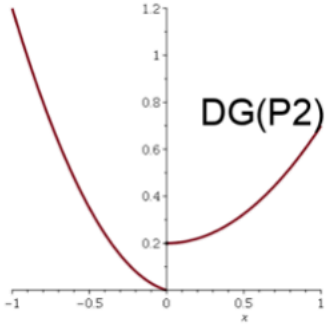
Discontinuous Galerkin (DG)

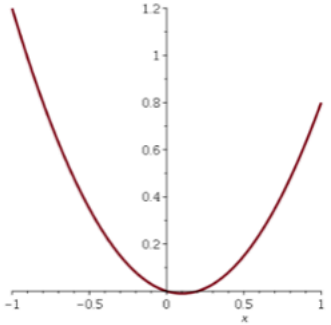
Finite Element (FE)

Continuous Galerkin (CG)









Discontinuous Galerkin (DG)

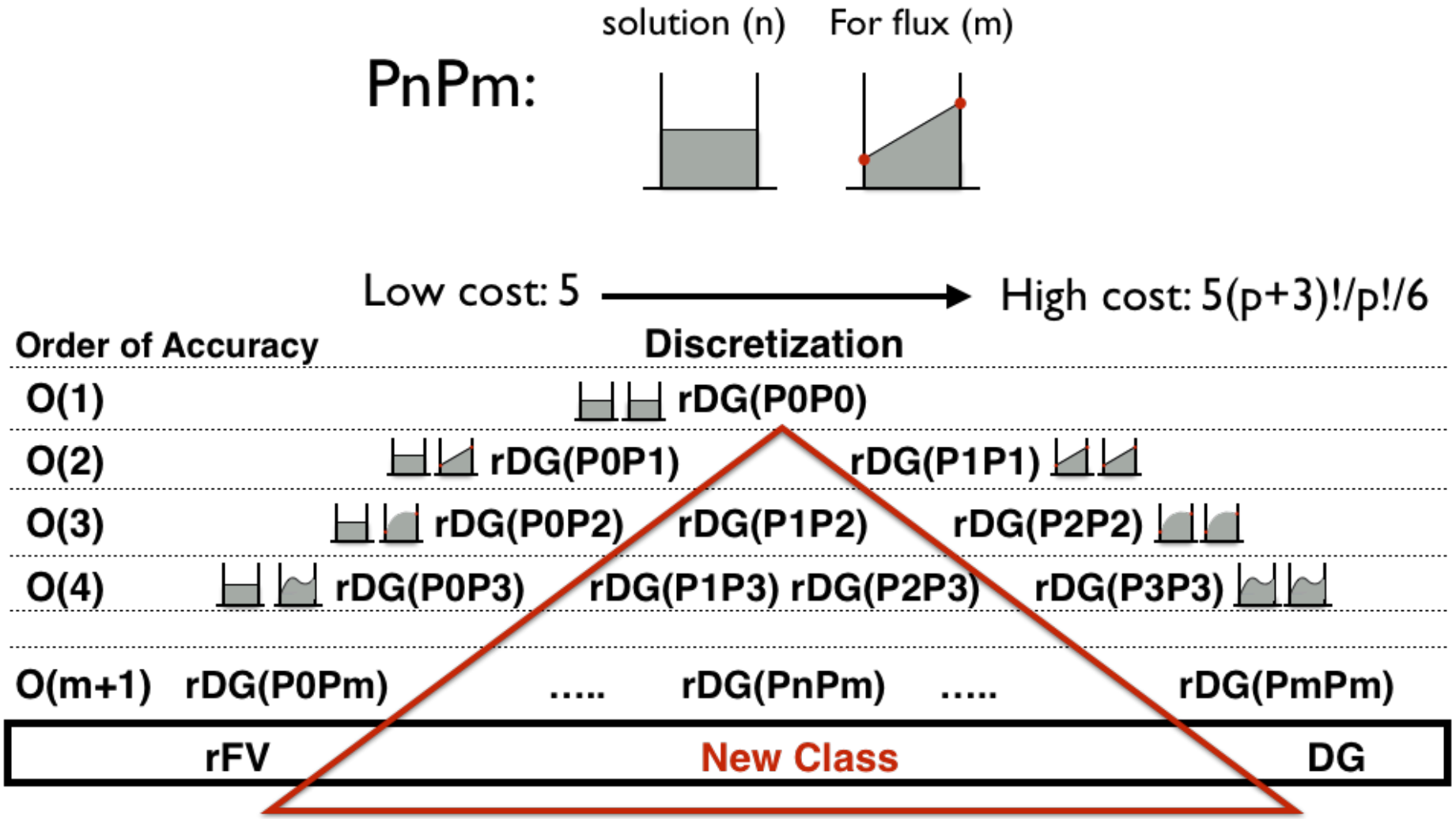
Pros

- High-order approximation with compact stencils
- Natural choice for hyperbolic problems
- Highly parallelizable

Cons

- Storage requirement
- Parabolic and elliptic problems

Reconstructed Discontinuous Galerkin (rDG)

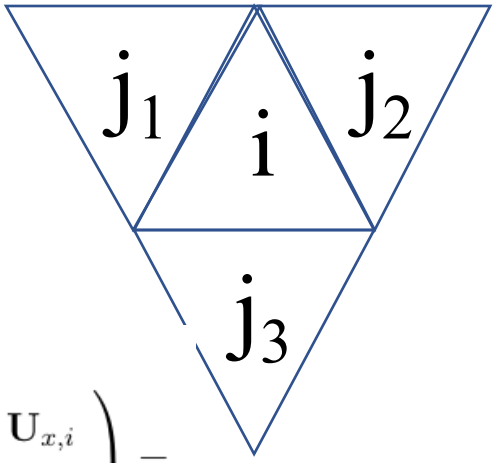


Accurate solution without significant increase in storage.

Reconstruction Method

Least-Squares Reconstruction

$$\begin{pmatrix} (x_{j1}-x_i) & (y_{j1}-y_i) \\ (x_{j2}-x_i) & (y_{j2}-y_i) \\ (x_{j3}-x_i) & (y_{j3}-y_i) \end{pmatrix} \begin{pmatrix} \mathbf{U}_{x,i} \\ \mathbf{U}_{y,i} \end{pmatrix} = \begin{pmatrix} \mathbf{U}_{j1}-\mathbf{U}_i \\ \mathbf{U}_{j2}-\mathbf{U}_i \\ \mathbf{U}_{j3}-\mathbf{U}_i \end{pmatrix}.$$



$$\begin{pmatrix} (x_{j1}-x_i) & (x_{j2}-x_i) & (x_{j3}-x_i) \\ (y_{j1}-y_i) & (y_{j2}-y_i) & (y_{j3}-y_i) \end{pmatrix} \begin{pmatrix} (x_{j1}-x_i) & (y_{j1}-y_i) \\ (x_{j2}-x_i) & (y_{j2}-y_i) \\ (x_{j3}-x_i) & (y_{j3}-y_i) \end{pmatrix} \begin{pmatrix} \mathbf{U}_{x,i} \\ \mathbf{U}_{y,i} \end{pmatrix} =$$

$$\begin{pmatrix} (x_{j1}-x_i) & (x_{j2}-x_i) & (x_{j3}-x_i) \\ (y_{j1}-y_i) & (y_{j2}-y_i) & (y_{j3}-y_i) \end{pmatrix} \begin{pmatrix} \mathbf{U}_{j1}-\mathbf{U}_i \\ \mathbf{U}_{j2}-\mathbf{U}_i \\ \mathbf{U}_{j3}-\mathbf{U}_i \end{pmatrix}.$$

Reconstruction Method

Variational Reconstruction

Solution to be reconstructed:

$$\begin{aligned}\mathbf{U}_i^R &= \tilde{\mathbf{U}}_i + \mathbf{U}_{x,i}\tilde{B}_2 + \mathbf{U}_{y,i}\tilde{B}_3 + \mathbf{U}_{z,i}\tilde{B}_4 \\ &+\mathbf{U}_{xx,i}\tilde{B}_5 + \mathbf{U}_{yy,i}\tilde{B}_6 + \mathbf{U}_{zz,i}\tilde{B}_7 + \mathbf{U}_{xy,i}\tilde{B}_8 + \mathbf{U}_{xz,i}\tilde{B}_9 + \mathbf{U}_{yz,i}\tilde{B}_{10}\end{aligned}$$

Minimize solution jumps at faces:

$$\mathbf{I} = \sum_{iface=1}^{nface} \mathbf{I}_{iface}$$

$$\mathbf{I}_{iface} = \frac{1}{d_{ij}} \int_{\Gamma_{ij}} \sum_{p=0}^{m} \sum_{k=0}^{p} \sum_{l=0}^{k} \left[\frac{\partial^p \mathbf{U}^-}{\partial x^l \partial y^{k-l} \partial z^{p-k}} - \frac{\partial^p \mathbf{U}^+}{\partial x^l \partial y^{k-l} \partial z^{p-k}} \right]^2 w_{p,k,l}^2 (d_{ij})^{2p} d\Gamma.$$

Reconstruction Method

Variational Reconstruction

Gradients are computed by solving a linear system:

$$\mathbf{M}_{jj}\mathbf{g}_j + \sum_{k\in\{k_j\}} \mathbf{M}_{jk}\mathbf{g}_k = RHS$$

P0P1-VR: 2x2 block (in 2D):

$$\mathbf{g}_j = [\partial_x u_j, \partial_y u_j]$$

P0P2-VR: 5x5 block (in 2D):

$$\mathbf{g}_j = [\partial_x u_j, \partial_y u_j, \partial_{xx} u_j, \partial_{xy} u_j, \partial_{yy} u_j]$$

The system is solved by the Gauss-Seidel relaxation scheme.
Advantage: Each step is compact and efficient, stabilizes FV/rDG solvers.

First Order Hyperbolic System (FOHS)

Advantages

- High-order and noise free gradients on irregular grids
- Upgraded accuracy in advection/inviscid limit
- Convergence acceleration (Elimination of numerical stiffness due to second-order derivatives)

Applications

- Model equations (diffusion/advection-diffusion)
- Navier-Stokes equations (HNS20/HNS20G)

Objective

Hyperbolic Navier-Stokes based on rDG/rFV with Variational Reconstruction

- Accurate for both primitive variables and their gradients (even on irregular or skewed mesh).
- Same degree of freedom with DG(P_1), but easily extended to high-order.

20 discrete unknowns per cell for up to 4th-order in the inviscid terms and 3^{rd}-order in the gradients.

Governing Equations

Compressible Navier-Stokes (NS) Equations

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}_k}{\partial x_k} = \frac{\partial \mathbf{G}_k}{\partial x_k},$$

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho e \end{pmatrix}, \quad \mathbf{F}_x = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ u(\rho e + p) \end{pmatrix}, \quad \mathbf{F}_y = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho vw \\ v(\rho e + p) \end{pmatrix}, \quad \mathbf{F}_z = \begin{pmatrix} \rho w \\ \rho uw \\ \rho vw \\ \rho w^2 + p \\ w(\rho e + p) \end{pmatrix}$$

$$\mathbf{G}_x = \begin{pmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ \tau_{xz} \\ u\tau_{xx} + v\tau_{xy} + w\tau_{xz} - q_x \end{pmatrix}, \quad \mathbf{G}_y = \begin{pmatrix} 0 \\ \tau_{yx} \\ \tau_{yy} \\ \tau_{yz} \\ u\tau_{yx} + v\tau_{yy} + w\tau_{yz} - q_y \end{pmatrix}, \quad \mathbf{G}_z = \begin{pmatrix} 0 \\ \tau_{zx} \\ \tau_{zy} \\ \tau_{zz} \\ u\tau_{zx} + v\tau_{zy} + w\tau_{zz} - q_z \end{pmatrix}$$

$$\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} = -\frac{2}{3}\mu(\operatorname{div}\mathbf{v})\mathbf{I} + \mu\left(\operatorname{grad}\mathbf{v} + (\operatorname{grad}\mathbf{v})^T\right), \quad \begin{pmatrix} q_x \\ q_y \\ q_z \end{pmatrix} = -\frac{\mu}{Pr(\gamma - 1)}\operatorname{grad} T,$$

HNS20G

FOHS: Hyperbolic Navier-Stokes (HNS20G) Equations

Introduce the gradients of density, velocity and temperature as auxiliary variables. (New formulation)

$$\mathbf{r} = \nabla\rho = (\rho_x, \rho_y, \rho_z),$$

$$[\mathbf{g}_u^T, \mathbf{g}_v^T, \mathbf{g}_w^T]^T = [(\nabla u)^T, (\nabla v)^T, (\nabla w)^T]^T = \begin{pmatrix} g_{ux} & g_{uy} & g_{uz} \\ g_{vx} & g_{vy} & g_{vz} \\ g_{wx} & g_{wy} & g_{wz} \end{pmatrix}$$

$$\mathbf{h} = \nabla T = (h_x, h_y, h_z).$$

which allows us to obtain higher-order derivatives easily.

HNS20G Equations

FOHS: Hyperbolic Navier-Stokes (HNS20G) Equations

$$\mathbf{P}^{-1}\frac{\partial \mathbf{U}}{\partial \tau}+\frac{\partial \mathbf{F}_k}{\partial x_k}=\mathbf{S}$$

$$\mathbf{U}=[\rho,\rho\mathbf{v},\rho e,\mathbf{g}_u,\mathbf{g}_v,\mathbf{g}_w,\mathbf{h},\mathbf{r}]^T,$$

$$\mathbf{F}=\begin{pmatrix}\rho\mathbf{v}^T-\nu_\rho\mathbf{r}^T\\ \rho\mathbf{v}\otimes\mathbf{v}+p\mathbf{I}-\mu_v\widetilde{\boldsymbol{\tau}}\\ \mathbf{v}^T(\rho e+p)-(\mu_v\widetilde{\boldsymbol{\tau}}\mathbf{v})^T-\dfrac{\mu_h}{\gamma(\gamma-1)}\mathbf{h}\\ -u\mathbf{I}\\ -v\mathbf{I}\\ -w\mathbf{I}\\ -T\mathbf{I}\\ -\rho\mathbf{I}\end{pmatrix}$$

$$\widetilde{\boldsymbol{\tau}}=\frac{3}{4}(\mathbf{g}+\mathbf{g}^T)-\frac{1}{2}tr(\mathbf{g})\mathbf{I},\quad \mu_v=\frac{4}{3}\mu,\quad \mu_h=\frac{\gamma}{Pr}\mu,$$

$$\mathbf{S}=[0\quad\mathbf{0}\quad 0\quad -\mathbf{g}_u\quad -\mathbf{g}_v\quad -\mathbf{g}_w\quad -\mathbf{h}\quad -\mathbf{r}]^T$$

$$\mathbf{P}=diag\left(1,1,1,1,1,\frac{1}{T_v},\frac{1}{T_v},\frac{1}{T_v},\frac{1}{T_v},\frac{1}{T_v},\frac{1}{T_v},\frac{1}{T_v},\frac{1}{T_v},\frac{1}{T_v},\frac{1}{T_h},\frac{1}{T_h},\frac{1}{T_h},\frac{1}{T_r},\frac{1}{T_r},\frac{1}{T_r}\right),$$

$$\mathbf{T}=diag\left(1,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0\right).$$

HNS20G+FV

Finite volume (FV) Method

$$\mathbf{P}^{-1}\frac{\partial \mathbf{U}}{\partial \tau}+\frac{\partial \mathbf{F}_k}{\partial x_k}=\mathbf{S}$$

$$\mathbf{U}\ (x,y,z,\tau)=\mathbf{CV}(\tau),$$

$$\int_{\Omega_e}\mathbf{P}^{-1}\frac{d\mathbf{U}}{d\tau}d\Omega=\int_{\Omega_e}\mathbf{S}\,(\mathbf{U}\ )\,d\Omega-\int_{\Gamma_e}\mathbf{F}_k\,(\mathbf{U}\ )\,n_k d\Gamma,$$

$$\mathbf{C}=diag(1,1,1,1,1,\frac{1}{\Delta x},\frac{1}{\Delta y},\frac{1}{\Delta z},\frac{1}{\Delta x},\frac{1}{\Delta y},\frac{1}{\Delta z},$$
$$\frac{1}{\Delta x},\frac{1}{\Delta y},\frac{1}{\Delta z},\frac{1}{\Delta x},\frac{1}{\Delta y},\frac{1}{\Delta z},\frac{1}{\Delta x},\frac{1}{\Delta y},\frac{1}{\Delta z}).$$

$$\begin{pmatrix}\rho\\ \rho u\\ \rho v\\ \rho w\\ \rho e\\ g_{ux}\\ g_{uy}\\ g_{uz}\\ g_{vx}\\ g_{vy}\\ g_{vz}\\ g_{wx}\\ g_{wy}\\ g_{wz}\\ h_x\\ h_y\\ h_z\\ r_x\\ r_y\\ r_z\end{pmatrix}=\mathbf{C}\begin{pmatrix}\overline{\rho}\\ \overline{\rho u}\\ \overline{\rho v}\\ \overline{\rho w}\\ \overline{\rho e}\\ \Delta x\overline{g}_{ux}\\ \Delta y\overline{g}_{uy}\\ \Delta z\overline{g}_{uz}\\ \Delta x\overline{g}_{vx}\\ \Delta y\overline{g}_{vy}\\ \Delta z\overline{g}_{vz}\\ \Delta x\overline{g}_{wx}\\ \Delta y\overline{g}_{wy}\\ \Delta z\overline{g}_{wz}\\ \Delta x\overline{h}_x\\ \Delta y\overline{h}_y\\ \Delta z\overline{h}_z\\ \Delta x\overline{r}_x\\ \Delta y\overline{r}_y\\ \Delta z\overline{r}_z\end{pmatrix}.$$

HNS20G+DG

DG Method With Full Galerkin Matrix

First rewrite the governing equation by introducing $\mathbf{W}$:

$$\mathbf{P}^{-1}\frac{\partial \mathbf{U}}{\partial \tau}+\frac{\partial \mathbf{F}_k}{\partial x_k}=\mathbf{S} \quad \Rightarrow \quad \mathbf{P}^{-1}\frac{\partial \mathbf{U}}{\partial \mathbf{W}}\frac{\partial \mathbf{W}}{\partial \tau}+\frac{\partial \mathbf{F}_k}{\partial x_k}=\mathbf{S}$$

We begin by formulating the discontinuous Galerkin discretization with a test function $\mathbf{\Psi}$ over an element Ω_e with a boundary Γ_e: multiply the HNS20G system by $\mathbf{\Psi}$, and integrate by parts over the element,

$$\frac{d}{d\tau}\int_{\Omega_e}\mathbf{\Psi}\mathbf{P}^{-1}\frac{\partial \mathbf{U}}{\partial \mathbf{W}}\mathbf{W}_h d\Omega=\int_{\Omega_e}\left[\frac{\partial \mathbf{\Psi}}{\partial x_k}\mathbf{F}_k(\mathbf{W}_h)+\mathbf{\Psi}\mathbf{S}(\mathbf{W}_h)\right]d\Omega-\int_{\Gamma_e}\mathbf{\Psi}\mathbf{F}_k(\mathbf{W}_h)n_k d\Gamma, \quad \forall \mathbf{\Psi}\in V_h^p.$$

where

$$\mathbf{W}=[\rho \quad \mathbf{v} \quad T \quad \mathbf{g}_u \quad \mathbf{g}_v \quad \mathbf{g}_w \quad \mathbf{h} \quad \mathbf{r}]^T$$

Now express the solution as a polynomial:

$$\mathbf{\Psi}=\mathbf{C}^{\mathbf{T}},$$

$$\mathbf{W}\ (x,y,z,\tau)=\mathbf{C}(x,y,z)\mathbf{V}(\tau),$$

DG Method With Full Galerkin Matrix

$$\mathbf{W}_h = (\rho, \mathbf{v}, T, \mathbf{g}_x, \mathbf{g}_y, \mathbf{g}_z, h_x, h_y, h_z, r_x, r_y, r_z)^T,$$

$$\mathbf{V} = \left(\overline{\rho}, \overline{\mathbf{v}}, \overline{T}, \Delta x\overline{\mathbf{g}_x}, \Delta y\overline{\mathbf{g}_y}, \Delta z\overline{\mathbf{g}_z}, \Delta x\overline{h_x}, \Delta y\overline{h_y}, \Delta z\overline{h_z}, \Delta x\overline{r_x}, \Delta y\overline{r_y}, \Delta z\overline{r_z}\right)^T,$$

$$\mathbf{C} = \begin{bmatrix}
1 & & & & & & & & & \frac{x-x_c}{\Delta x} & \frac{y-y_c}{\Delta y} & \frac{z-z_c}{\Delta z} \\
 & 1 & & \frac{x-x_c}{\Delta x} & \frac{y-y_c}{\Delta y} & \frac{z-z_c}{\Delta z} & & & & & & \\
 & & 1 & & & & \frac{x-x_c}{\Delta x} & \frac{y-y_c}{\Delta y} & \frac{z-z_c}{\Delta z} & & & \\
 & & & \frac{1}{\Delta x} & & & & & & & & \\
 & & & & \frac{1}{\Delta y} & & & & & & & \\
 & & & & & \frac{1}{\Delta z} & & & & & & \\
 & & & & & & \frac{1}{\Delta x} & & & & & \\
 & & & & & & & \frac{1}{\Delta y} & & & & \\
 & & & & & & & & \frac{1}{\Delta z} & & & \\
 & & & & & & & & & \frac{1}{\Delta x} & & \\
 & & & & & & & & & & \frac{1}{\Delta y} & \\
 & & & & & & & & & & & \frac{1}{\Delta z}
\end{bmatrix}.$$

$$\mathbf{M}_p \frac{d\mathbf{V}}{d\tau} = \mathbf{R}(\mathbf{W}),$$

$$\mathbf{M}_p = \int_{\Omega_e} \mathbf{C}^T \mathbf{P}^{-1} \frac{\partial \mathbf{U}}{\partial \mathbf{W}} \mathbf{C}\, d\Omega, \quad \mathbf{R}(\mathbf{W}) = \int_{\Omega_e} \left[\frac{\partial \mathbf{C}^T}{\partial x_k}\mathbf{F}_k(\mathbf{V}) + \mathbf{C}^T\mathbf{S}(\mathbf{V})\right] d\Omega - \int_{\Gamma_e} \mathbf{C}^T \mathbf{F}_k(\mathbf{V}) n_k\, d\Gamma,$$

This is HNS(P0P1+P0), which is made higher-order with rDG ->

HNS20G+rDG

Reconstructed Discontinuous Galerkin Method

$$\mathbf{R}\left(\mathbf{W}^R\right) = \int_{\Omega_e} \left[\frac{\partial \mathbf{C}^T}{\partial x_k}\mathbf{F}_k(\mathbf{V}^R) + \mathbf{C}^T\mathbf{S}(\mathbf{V}^R)\right] d\Omega - \int_{\Gamma_e} \mathbf{C}^T\mathbf{F}_k(\mathbf{V}^R)n_k\, d\Gamma.$$

$$\begin{pmatrix} \rho \\ r_x \\ r_y \\ r_z \end{pmatrix}_h^R = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 & B_8 & B_9 & B_{10} \\ 0 & \frac{1}{\Delta x} & 0 & 0 & \frac{B_2}{\Delta x} & 0 & 0 & \frac{B_3}{\Delta x} & \frac{B_4}{\Delta x} & 0 \\ 0 & 0 & \frac{1}{\Delta y} & 0 & 0 & \frac{B_3}{\Delta y} & 0 & \frac{B_2}{\Delta y} & 0 & \frac{B_4}{\Delta x} \\ 0 & 0 & 0 & \frac{1}{\Delta z} & 0 & 0 & \frac{B_4}{\Delta z} & \frac{B_2}{\Delta z} & 0 & \frac{B_3}{\Delta z} \end{pmatrix} \begin{pmatrix} \overline{\rho} \\ \overline{r_x}\Delta x \\ \overline{r_y}\Delta y \\ \overline{r_z}\Delta z \\ r_{xx}^c\Delta x^2 \\ r_{yy}^c\Delta y^2 \\ r_{zz}^c\Delta z^2 \\ r_{xy}^c\Delta x\Delta y \\ r_{xz}^c\Delta x\Delta z \\ r_{yz}^c\Delta y\Delta z \end{pmatrix}.$$

Reconstructed(LSQ or VR)

HNS(P0P2-P0P1)-rDG: 3rd-order inviscid + 2nd-order viscous/gradient

4 unknowns per variable -> 20 unknowns per cell like P1-DG

but 1 order higher in the inviscid terms and gradients than P1-DG.

Reconstruction Method

Euler Equations

The reconstruction is applied to conservative variables.

$rDG(P_nP_m)$

HNS20G Equations

The reconstruction is applied to gradients.

$HNS(P_0P_{m+1}+P_0P_m)+rFV$

$HNS(P_0P_{m+1}+P_0P_m)+rDG$

Time Marching

Steady Problems

Fully implicit pseudo time-marching

Newton iteration

Unsteady Problems

3rd-order ESDIRK time integration method is implemented for physical time marching.

Newton iteration method is used at each stage.

Linear solver: both SGS(k) and Linearized GMRES are implemented at each Newton iteration.

Numerical Examples

Test for VR reconstruction

Case 1: K-exactness for Variational Reconstruction

Compressible Inviscid Flow (rDG(P_nP_m_VR))

Case 2: Subsonic Flow Past a Circular Cylinder

Case 3: Subsonic Flow Past a Sphere

Case 4: Transonic Flow Past an ONERA M6 Wing

Case 5: Transonic Flow Past a Boeing 747 Aircraft

Compressible Viscous Flow (HNS(P_0P_{m+1}+P_0P_m)+rFV)

Case 6: Manufactured Solution

Case 7: Laminar Flow Past a Flat Plate

Compressible Viscous Flow (HNS(P_0P_{m+1}+P_0P_m)+rDG)

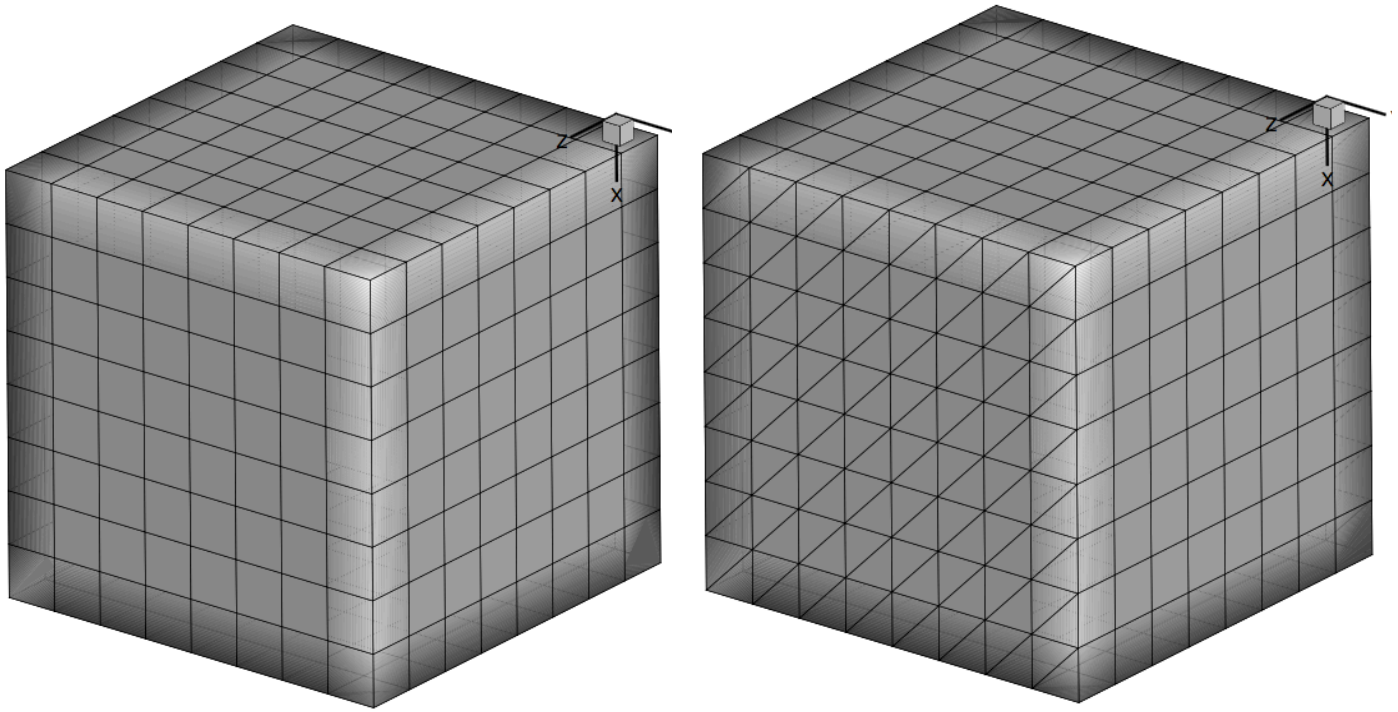
Case 8: Laminar Flow Past a Sphere

Case 9: Laminar Flow Past a Delta Wing

Case 10: Unsteady Laminar Flow Past a Circular Cylinder

Test for VR reconstruction

Case 1: K-exactness for Variational Reconstruction



Only gradients.

Test for VR reconstruction

Case 1: K-exactness on Hexahedral Mesh

Table 1: Order of convergence for test case 1 on hexahedral grids

	P_1P_2-LS		P_1P_2-VR	
N_e	L2-error	Order	L2-error	Order
$f(x,y,z) = \frac{1}{2}(999x^2 - 888y^2 + 777z^2) - 666xy + 555xz - 444yz$				
$8 \times 8 \times 8$	1.0153e-12	-	1.0141e-12	-
$16 \times 16 \times 16$	1.4110e-12	-	1.4010e-12	-
$32 \times 32 \times 32$	1.2570e-12	-	1.2577e-12	-
$f(x,y,z) = sin(\pi x)sin(\pi y)sin(\pi z)$				
$8 \times 8 \times 8$	9.4959e-4	-	1.1708e-3	-
$16 \times 16 \times 16$	1.0687e-4	3.152	1.2178e-4	3.265
$32 \times 32 \times 32$	1.2832e-5	3.058	1.3711e-5	3.151

Test for VR reconstruction

Case 1: K-exactness on Prismatic Mesh

Table 2: Order of convergence of test case 1 on prismatic grids

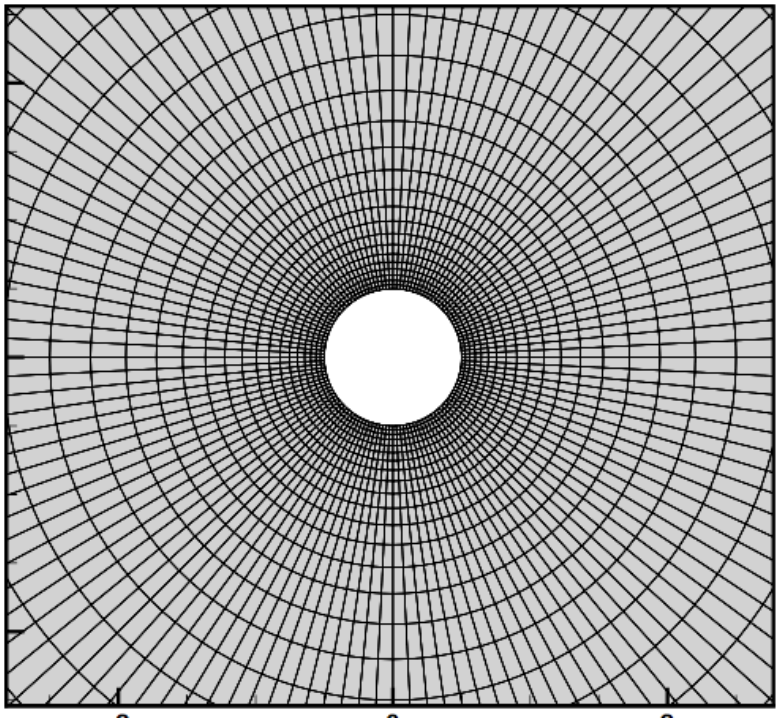
	P_1P_2_VR	
N_e	L2-error	Order
$f(x,y,z) = \frac{1}{2}(999x^2 - 888y^2 + 777z^2) - 666xy + 555xz - 444yz$		
$2 \times 8 \times 8 \times 8$	7.2165e-13	-
$2 \times 16 \times 16 \times 16$	1.2509e-12	-
$2 \times 32 \times 32 \times 32$	1.7187e-12	-
$f(x,y,z) = sin(\pi x)sin(\pi y)sin(\pi z)$		
$2 \times 8 \times 8 \times 8$	5.6690e-4	-
$2 \times 16 \times 16 \times 16$	5.8323e-5	3.281
$2 \times 32 \times 32 \times 32$	6.7368e-6	3.114

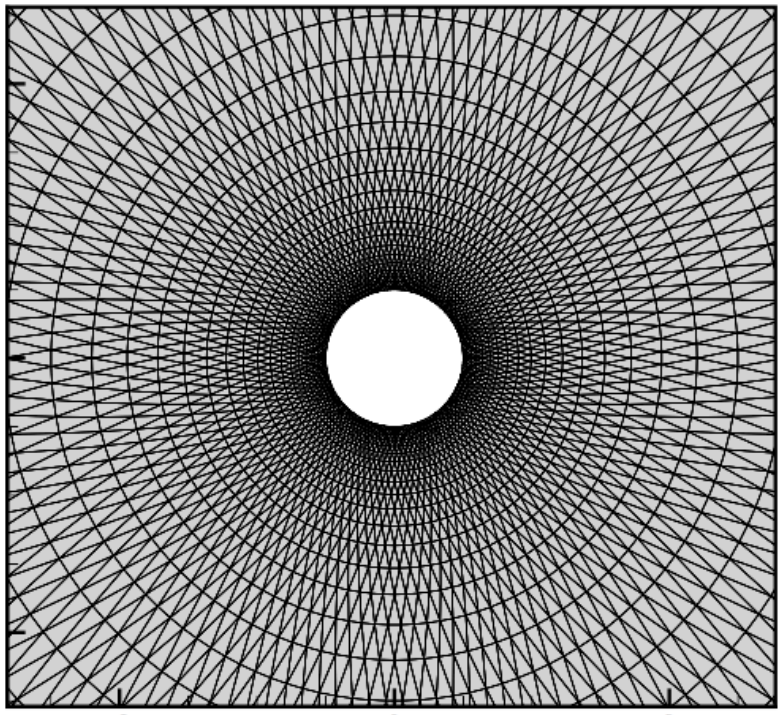
Compressible Inviscid Flow (rDG(P_nP_m_VR))

Case 2: Subsonic Flow Past a Circular Cylinder

Mach number = 0.38

Mesh points: $16\times5, 32\times9, 64\times17, 128\times33$

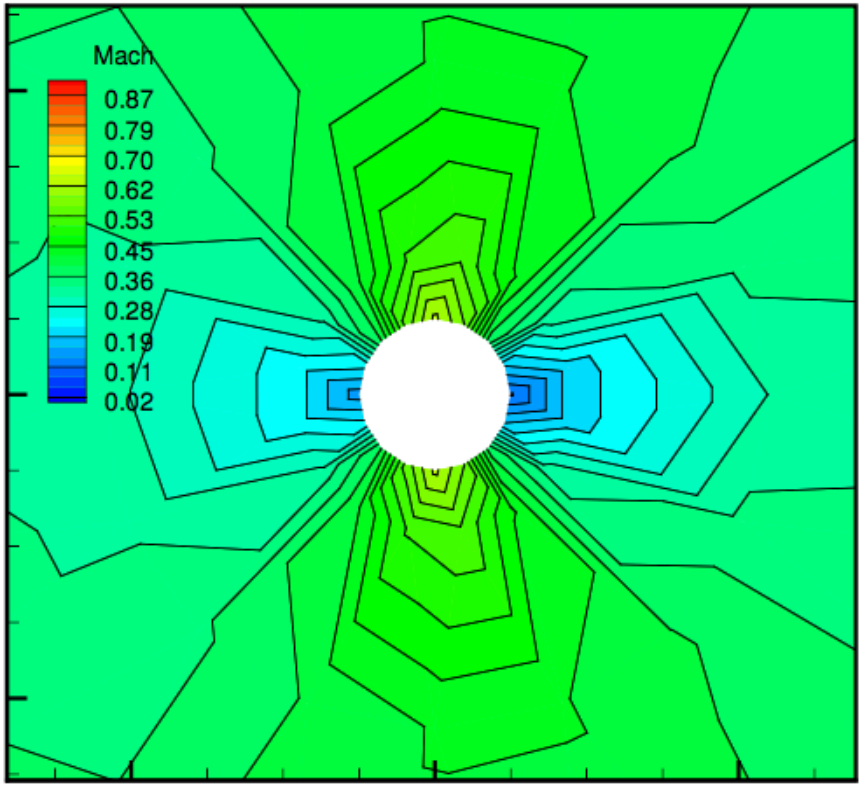


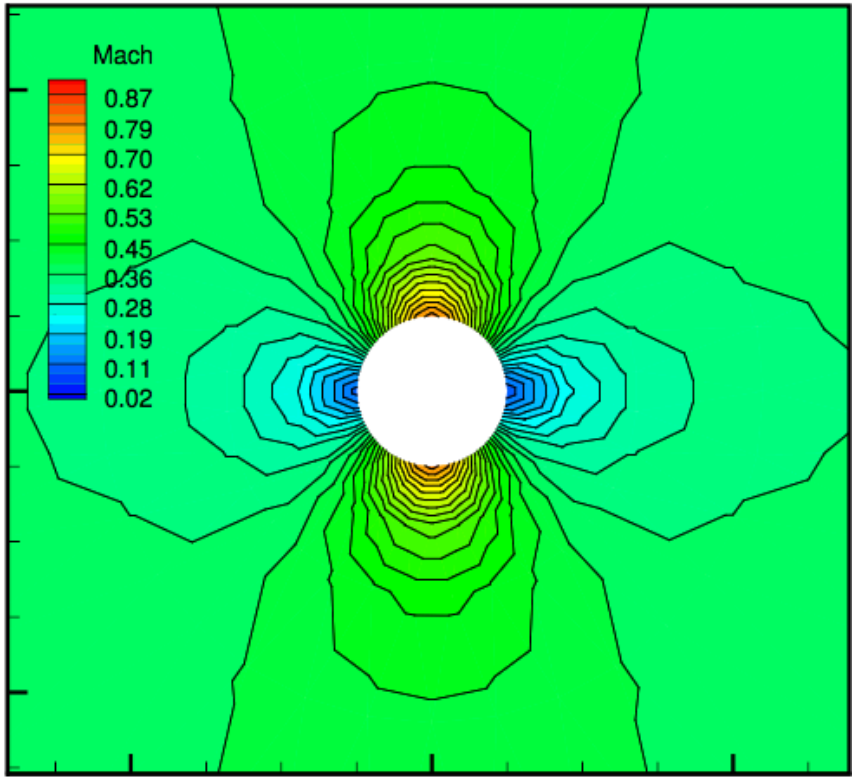


Compressible Inviscid Flow (rDG(P_nP_m_VR))

Case 2: Subsonic Flow Past a Circular Cylinder

Mach contours on prismatic mesh (rDG(P_1P_2_VR))

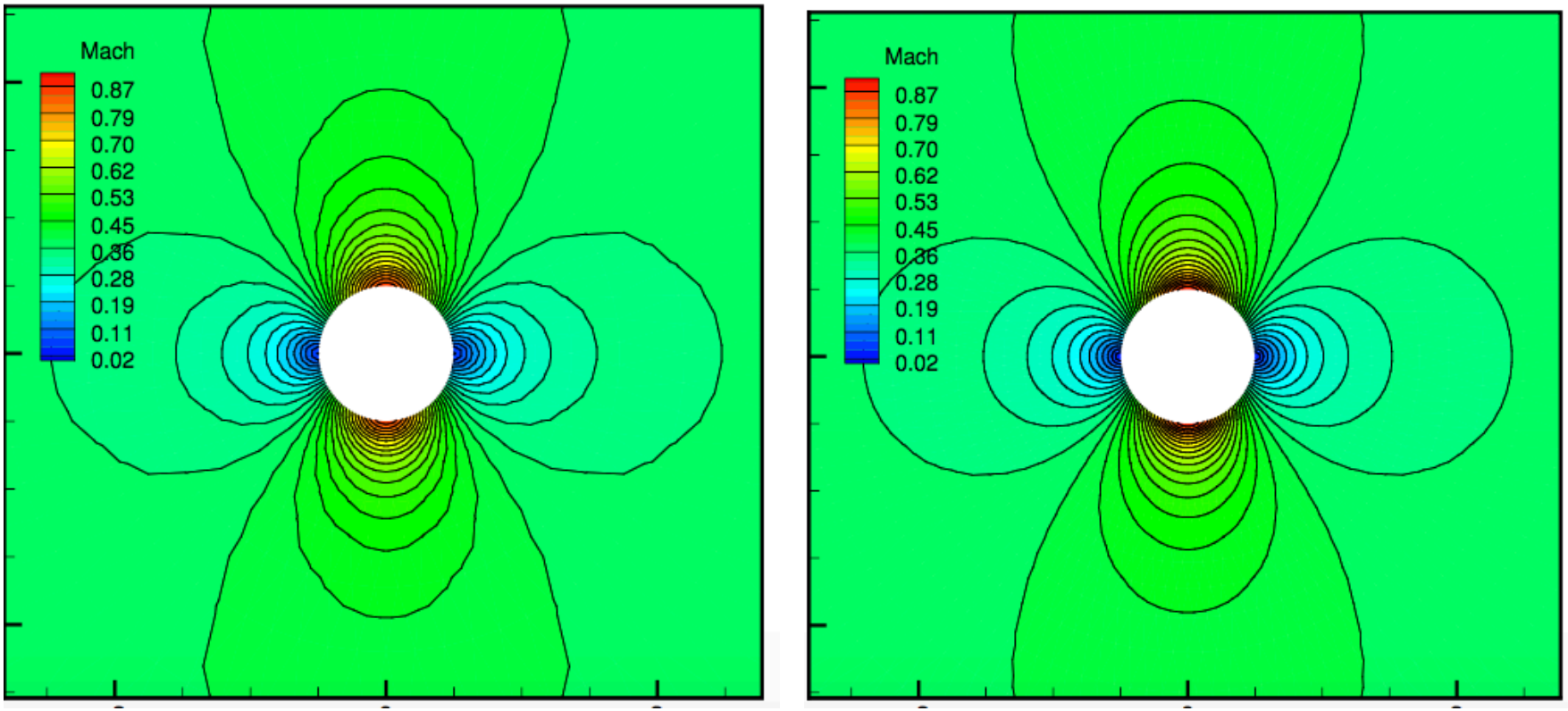




Compressible Inviscid Flow (rDG(P_nP_m_VR))

Case 2: Subsonic Flow Past a Circular Cylinder

Mach contours on prismatic mesh (rDG(P_1P_2_VR))



Compressible Inviscid Flow (rDG(P_nP_m_VR))

Case 2: Order of Convergence on Hexahedral Mesh

Table 3: Order of convergence for inviscid flow past a cylinder on hexahedral grids

	rDG(P_1P_2-VR)		rDG(P_1P_2-LS)	
$Log_{10}(h)$	$Log_{10}(error_{L2})$	Order	$Log_{10}(error_{L2})$	Order
-0.903	-0.8594	-	-0.8865	-
-1.204	-1.7138	2.839	-1.8036	3.047
-1.505	-2.9497	4.106	-3.0157	4.027
-1.806	-4.1987	4.150	-4.0653	3.487

Compressible Inviscid Flow (rDG(P_nP_m_VR))

Case 2: Order of Convergence on Prismatic Mesh

Table 4: Order of convergence for inviscid flow past a cylinder on prismatic grids

	rDG(P_1P_2_VR)		rDG(P_1P_2_LS)	
$Log_{10}(h)$	$Log_{10}(error_{L2})$	Order	$Log_{10}(error_{L2})$	Order
-1.054	-1.4768	-	-1.3742	-
-1.355	-2.6658	3.950	-2.5413	3.877
-1.657	-3.9742	4.347	-3.7026	3.858
-1.957	-5.2016	4.078	-4.4100	2.350

Compressible Inviscid Flow (rDG(P_nP_m_VR))

Case 3: Subsonic Flow Past a Sphere

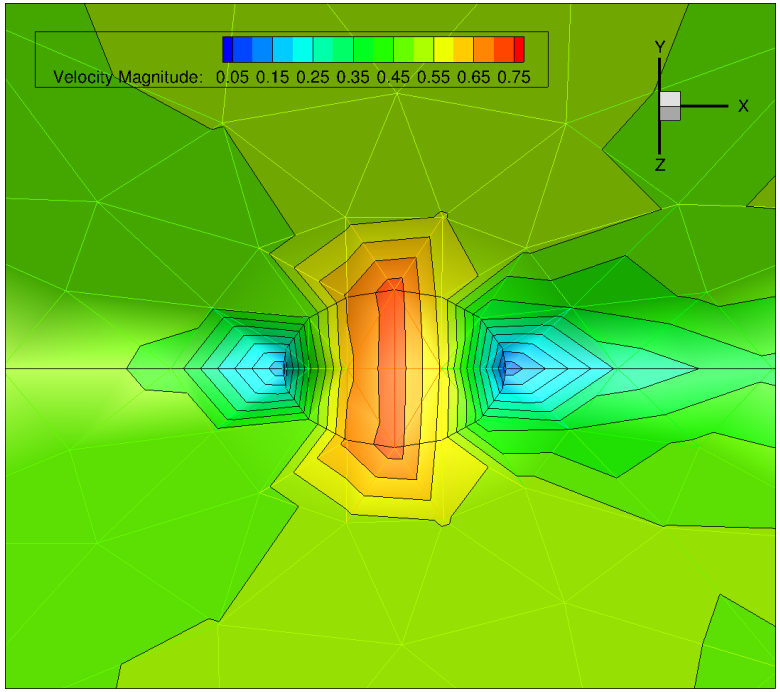
Mach number = 0.50

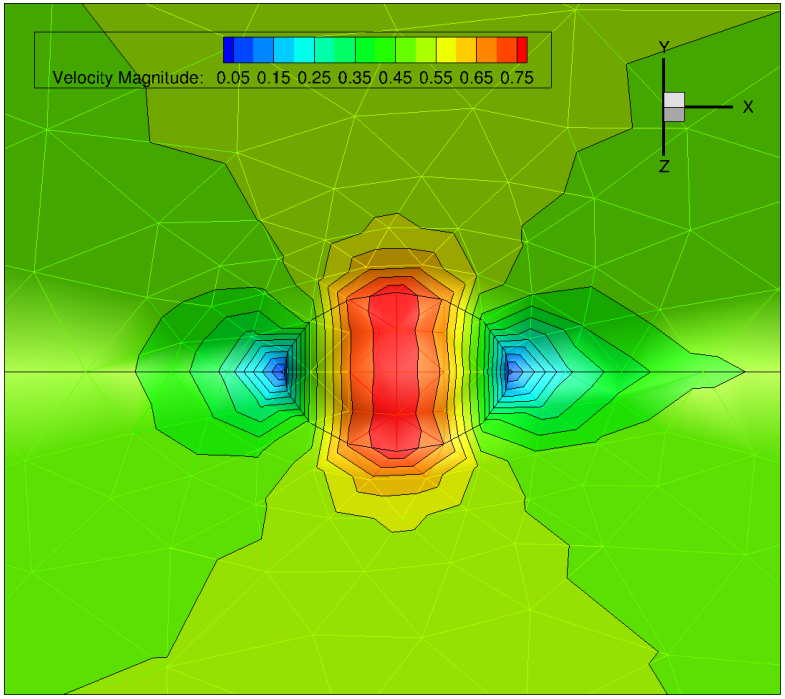
Mesh	Tetrahedral	Points	Boundary Faces
Level 1	535	167	124
Level 2	2426	598	322
Level 3	16467	3425	1188
Level 4	124706	176165	9072

Compressible Inviscid Flow (rDG(P_nP_m_VR))

Case 3: Subsonic Flow Past a Sphere

Contours of Velocity Magnitude

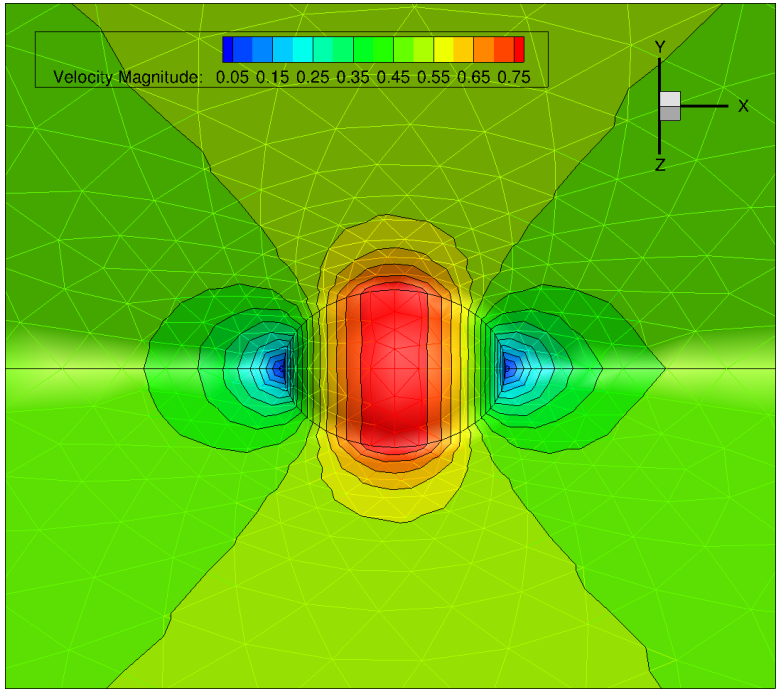


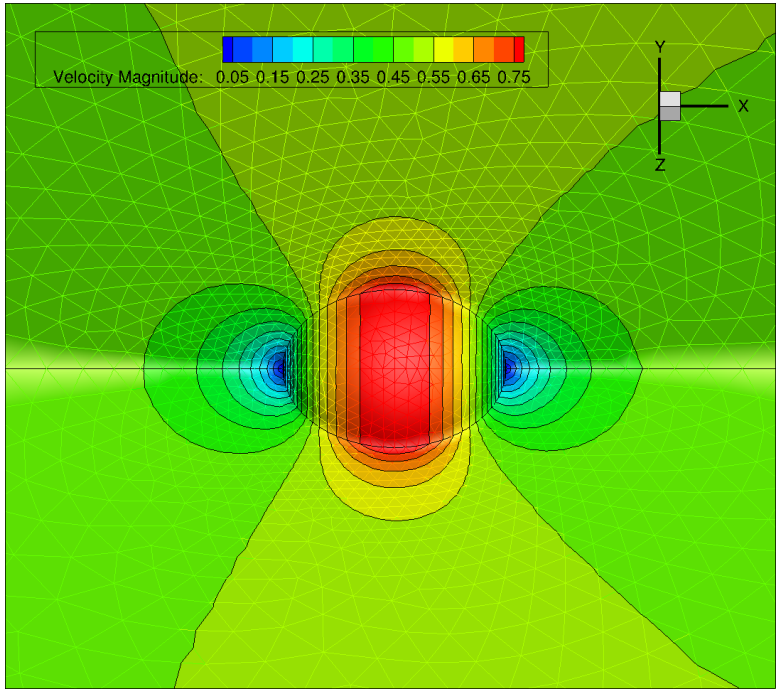


Compressible Inviscid Flow (rDG(P_nP_m_VR))

Case 3: Subsonic Flow Past a Sphere

Contours of Velocity Magnitude

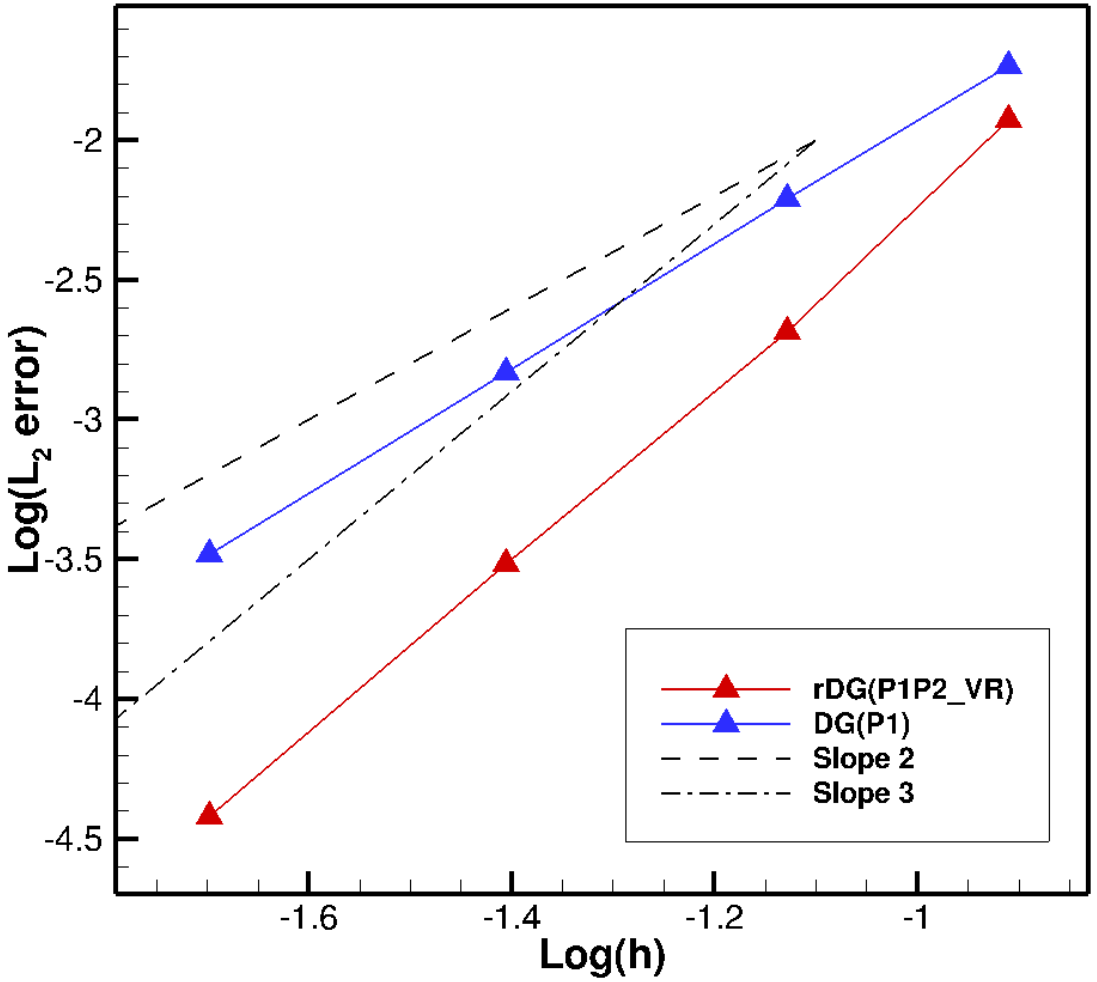




Compressible Inviscid Flow (rDG(P_nP_m_VR))

Case 3: Subsonic Flow Past a Sphere

Order of Convergence



Compressible Inviscid Flow (rDG(P_nP_m_VR))

Case 4: Transonic Flow Past an ONERA M6 Wing

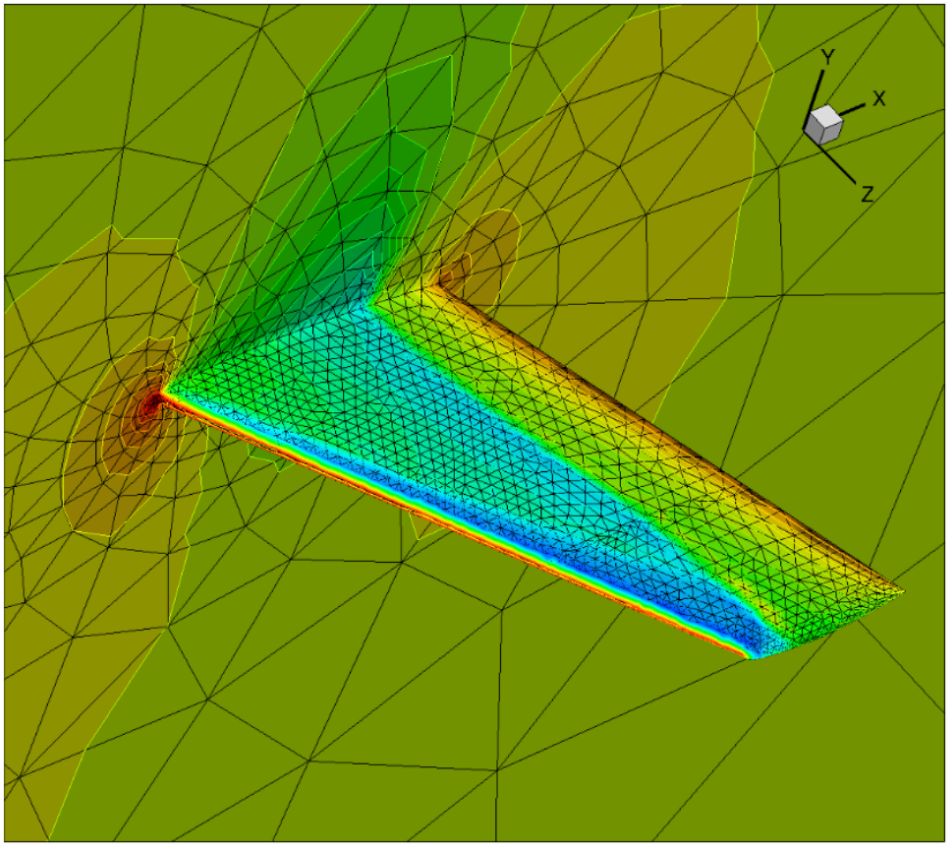
Mach number = 0.84

Angle of attack = 3.06°

Tetrahedral cells: 593,169

Grid points: 110,282

Boundary faces: 39,770



Mach contours with rDG(P_0P_1_VR)

Compressible Inviscid Flow (rDG(P_nP_m_VR))

Case 5: Transonic Flow Past a Boeing 747 aircraft

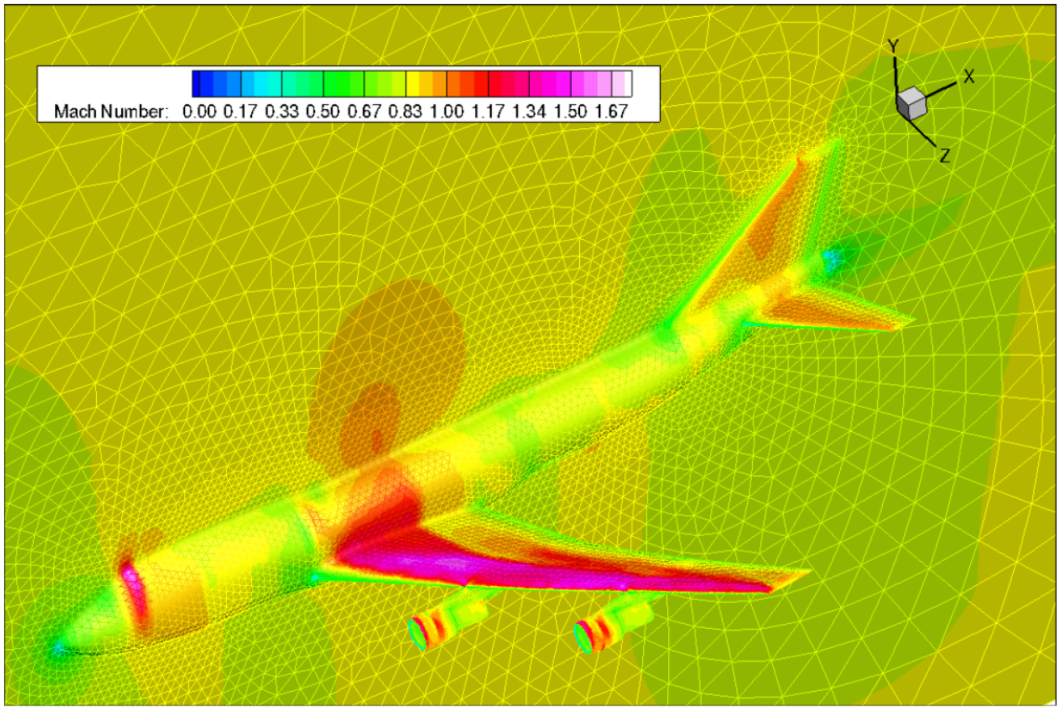
Mach number = 0.85

Angle of attack = 2.00°

Tetrahedral cells: 1,025,170

Grid points: 371,162

Boundary faces: 60,780

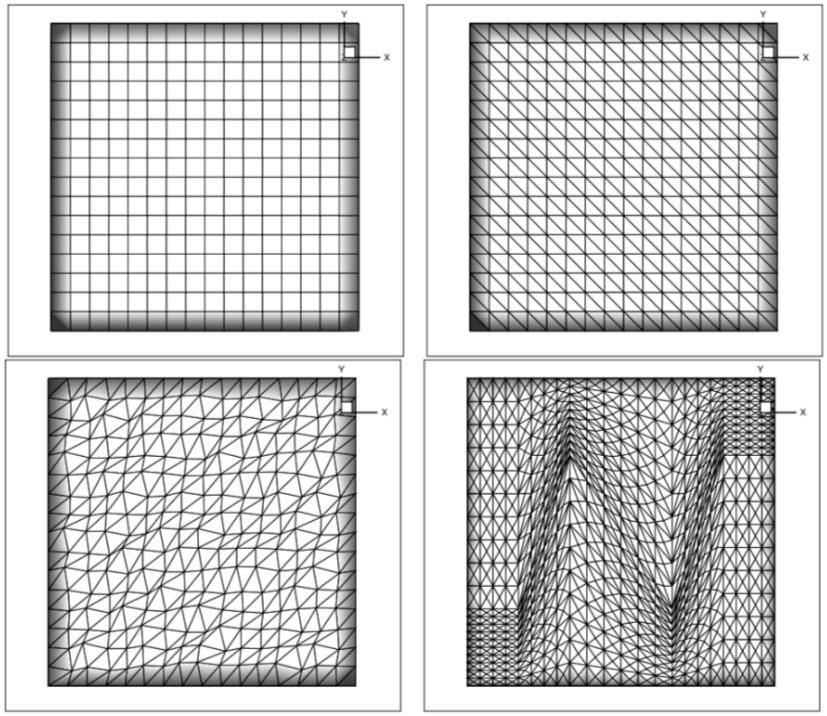


Mach Contours with rDG(P_0P_1_VR)

Compressible Viscous Flow (HNS(P_0P_{m+1}+P_0P_m)+rFV)

Case 6: Manufactured Solution (HNS+rFV)

$$\begin{cases} \rho = cr0 + crs \cdot sin(crx \cdot x + cry \cdot y), \\ u = cu0 + cus \cdot sin(cux \cdot x + cuy \cdot y), \\ v = cv0 + cvs \cdot sin(cvx \cdot x + cvy \cdot y), \\ w = 0, \\ p = cp0 + cps \cdot sin(cpx \cdot x + cpy \cdot y), \end{cases}$$



Compressible Viscous Flow (HNS(P_0P_{m+1}+P_0P_m)+rFV)

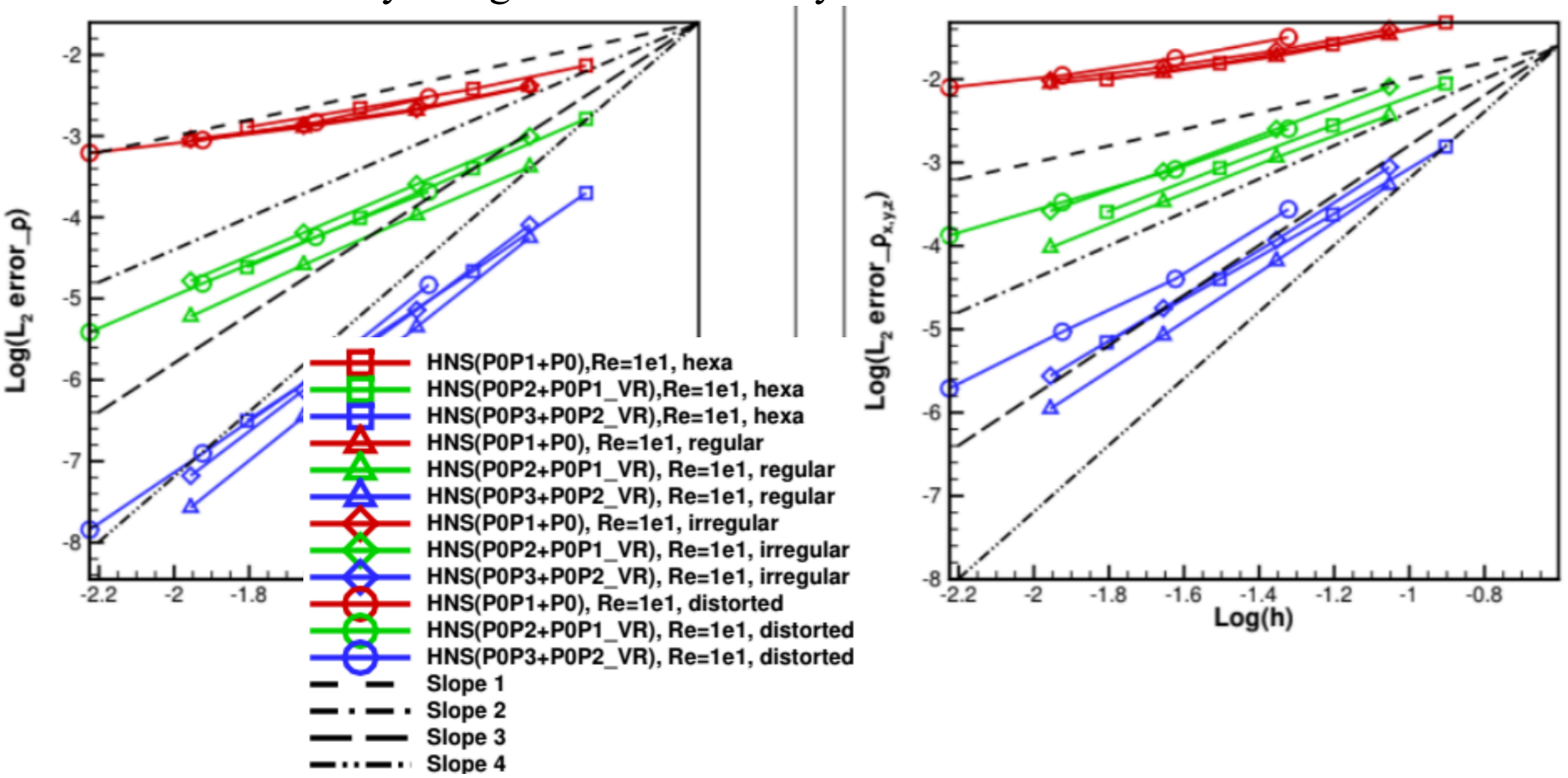
Case 6: Manufactured Solution (HNS+rFV)

Scheme	DoF/Cell	Inviscid	Viscous	Gradient
DG(P1)	20	$O(h^2)$	$O(h^2)$	$O(h)$
DG(P2)	50	$O(h^3)$	$O(h^3)$	$O(h^2)$
HNS(P0P1+P0)	20	$O(h^2)$	$O(h)$	$O(h)$
HNS(P0P2+P0P1(VR))	20	$O(h^3)$	$O(h^2)$	$O(h^2)$
HNS(P0P3+P0P2(VR))	20	$O(h^4)$	$O(h^3)$	$O(h^3)$

Compressible Viscous Flow (HNS(P_0P_{m+1}+P_0P_m)+rFV)

Case 6: Manufactured Solution (HNS+rFV)

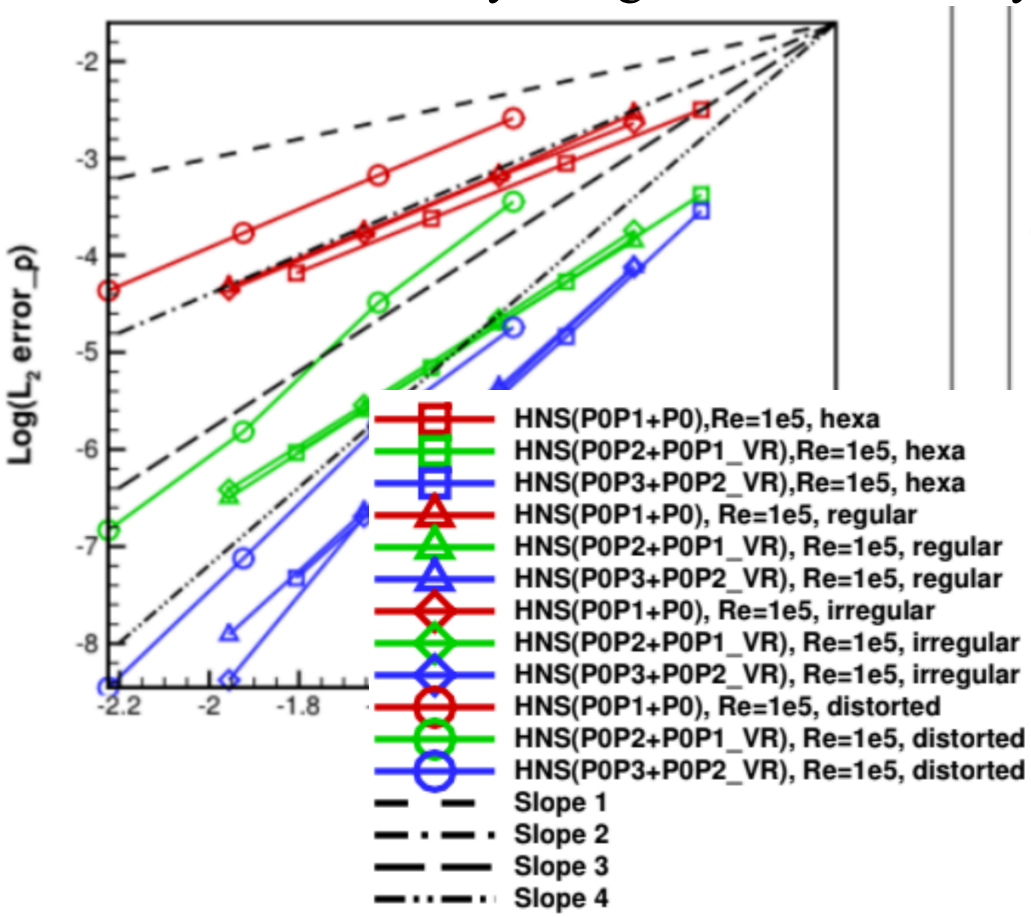
Re = 10. For density and gradients of density.

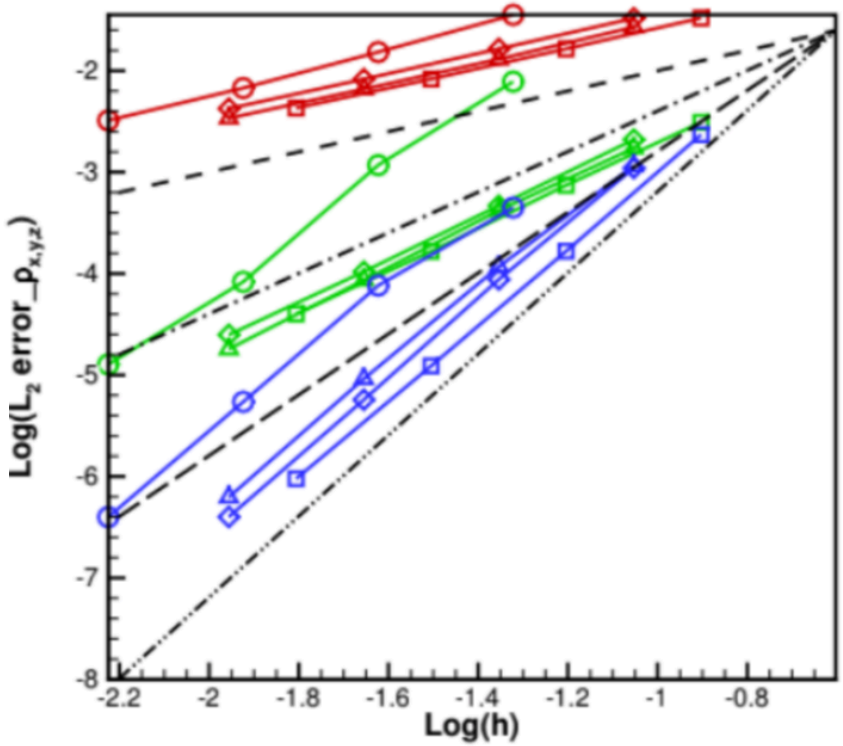


Compressible Viscous Flow (HNS(P_0P_{m+1}+P_0P_m)+rFV)

Case 6: Manufactured Solution (HNS+rFV)

Re = 1e5. For density and gradients of density.

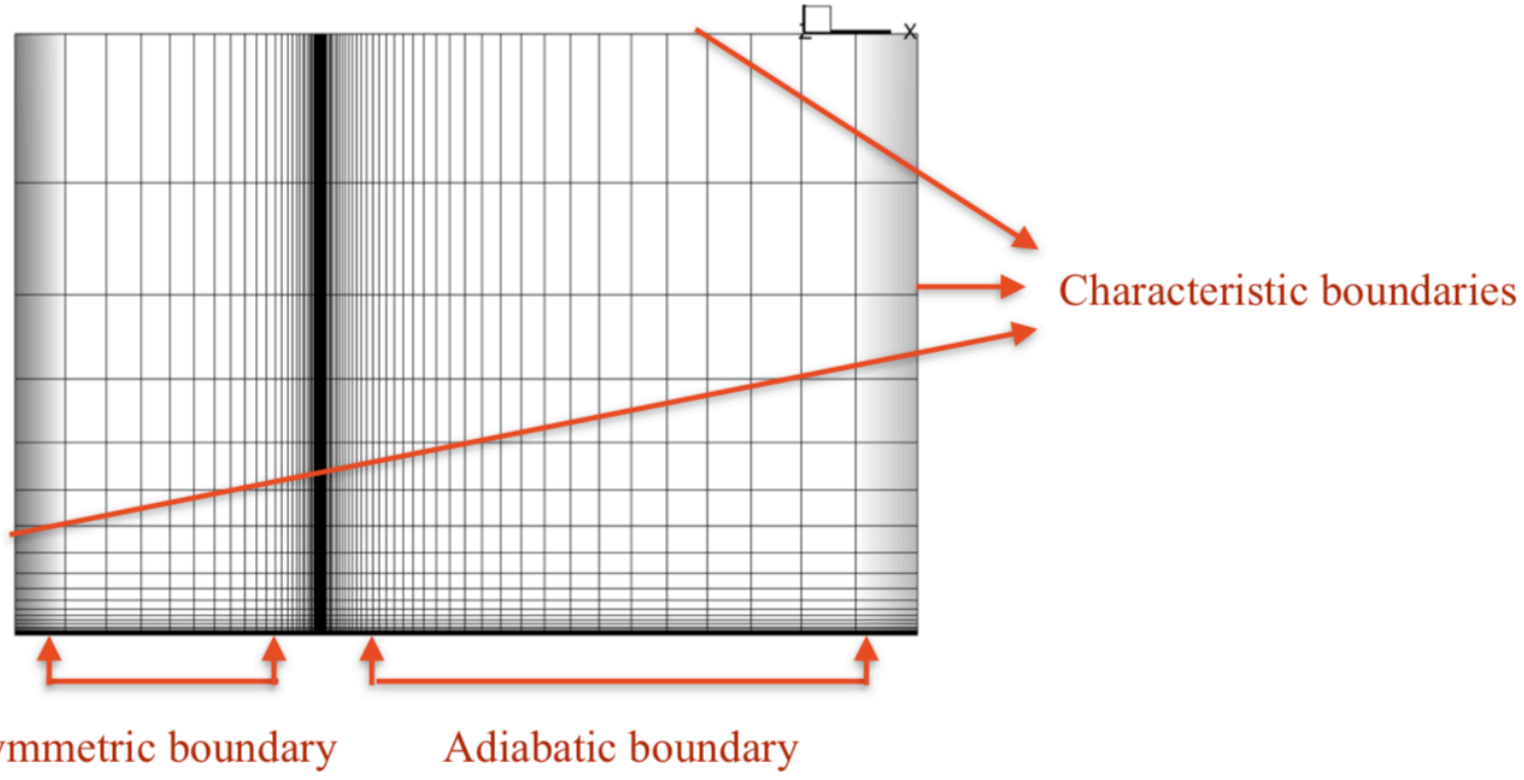




Compressible Viscous Flow (HNS(P_0P_{m+1}+P_0P_m)+rFV)

Case 7: Laminar Flow Past a Flat Plate (HNS+rFV)

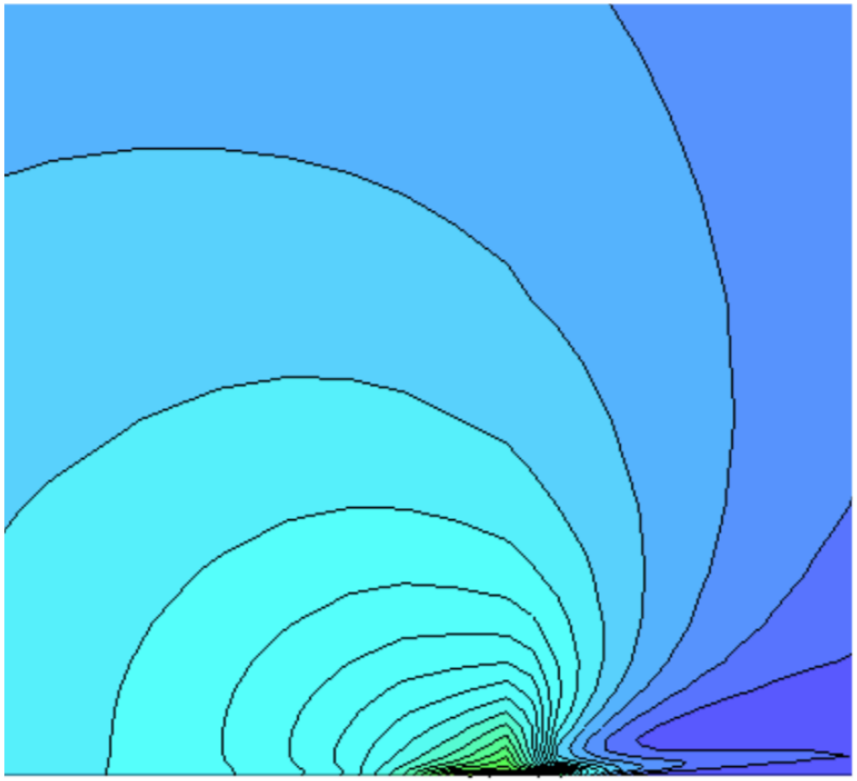
Mach = 0.2; Reynolds number =1e5.



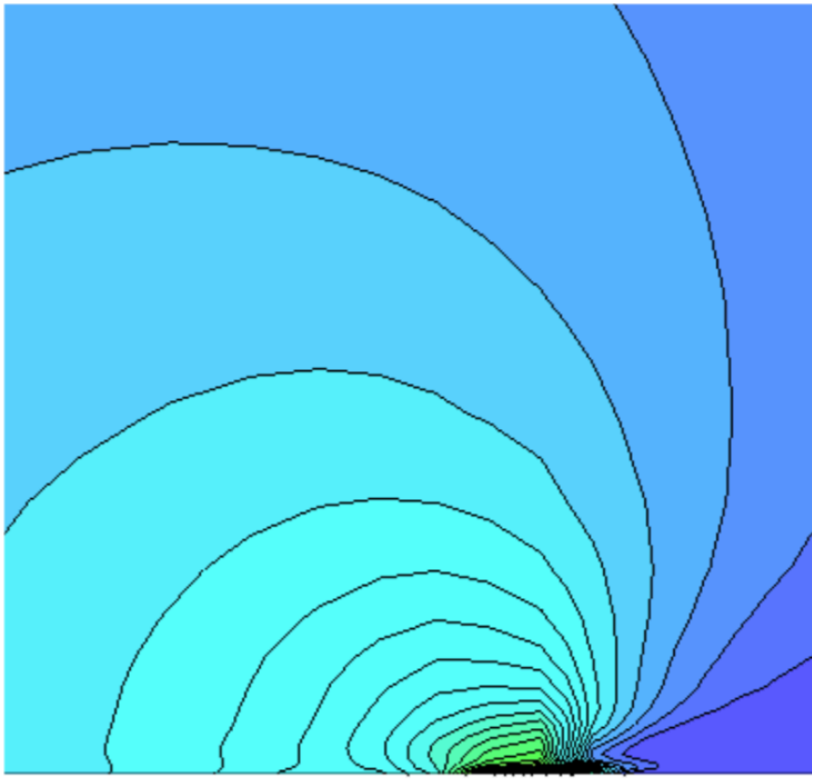
Compressible Viscous Flow (HNS(P_0P_{m+1}+P_0P_m)+rFV)

Case 7: Laminar Flow Past a Flat Plate (HNS+rFV)

Pressure contours



HNS(P_0P_1+P_0)+FV

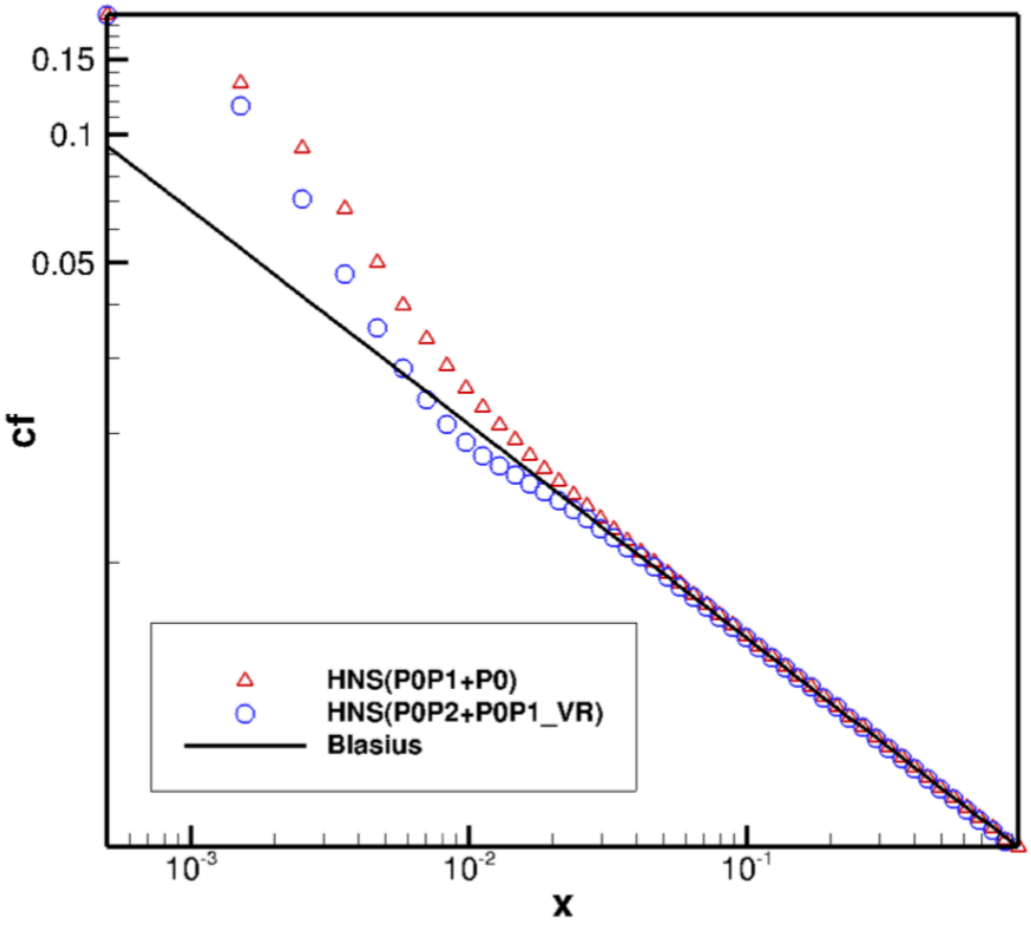


HNS(P_0P_2+P_0P_1_VR)+rFV

Compressible Viscous Flow (HNS(P_0P_{m+1}+P_0P_m)+rFV)

Case 7: Laminar Flow Past a Flat Plate (HNS+rFV)

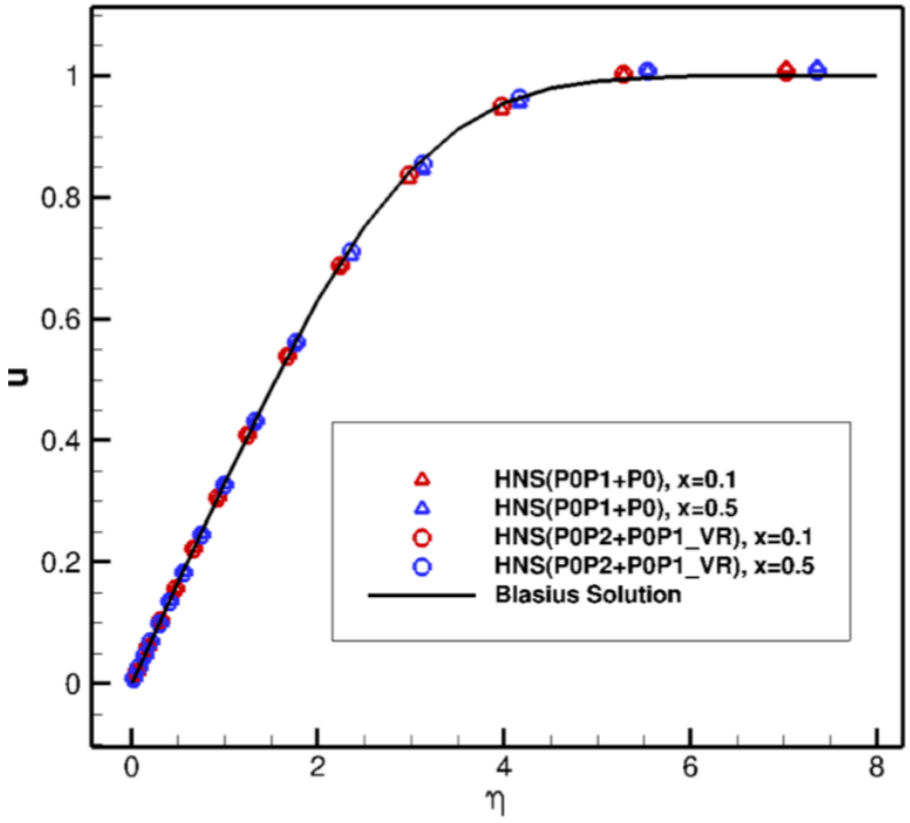
Skin friction

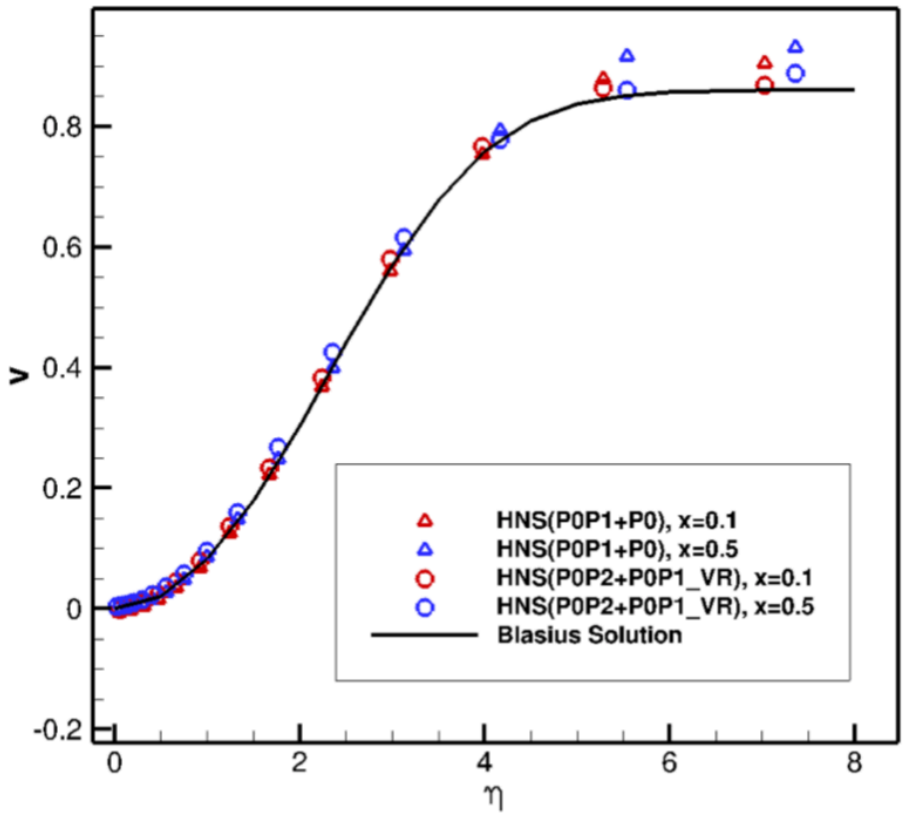


Compressible Viscous Flow (HNS(P_0P_{m+1}+P_0P_m)+rFV)

Case 7: Laminar Flow Past a Flat Plate (HNS+rFV)

Velocity distribution.





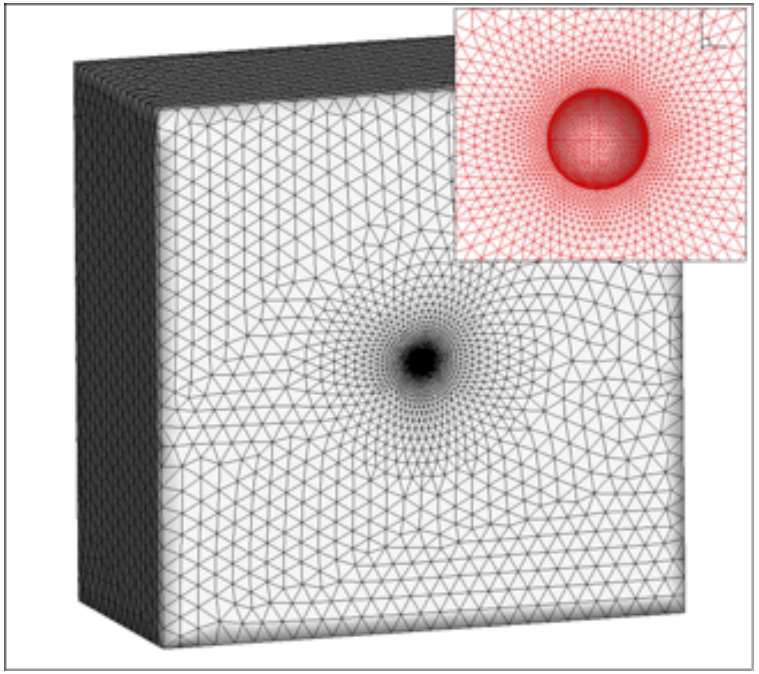
Compressible Viscous Flow (HNS(P_0P_{m+1}+P_0P_m)+rDG)

Case 8: Laminar Flow Past a Sphere (HNS+rDG)

- Mach = 0.5
- Re=100
- Number of tetrahedral:

  200,416
- Number of points:

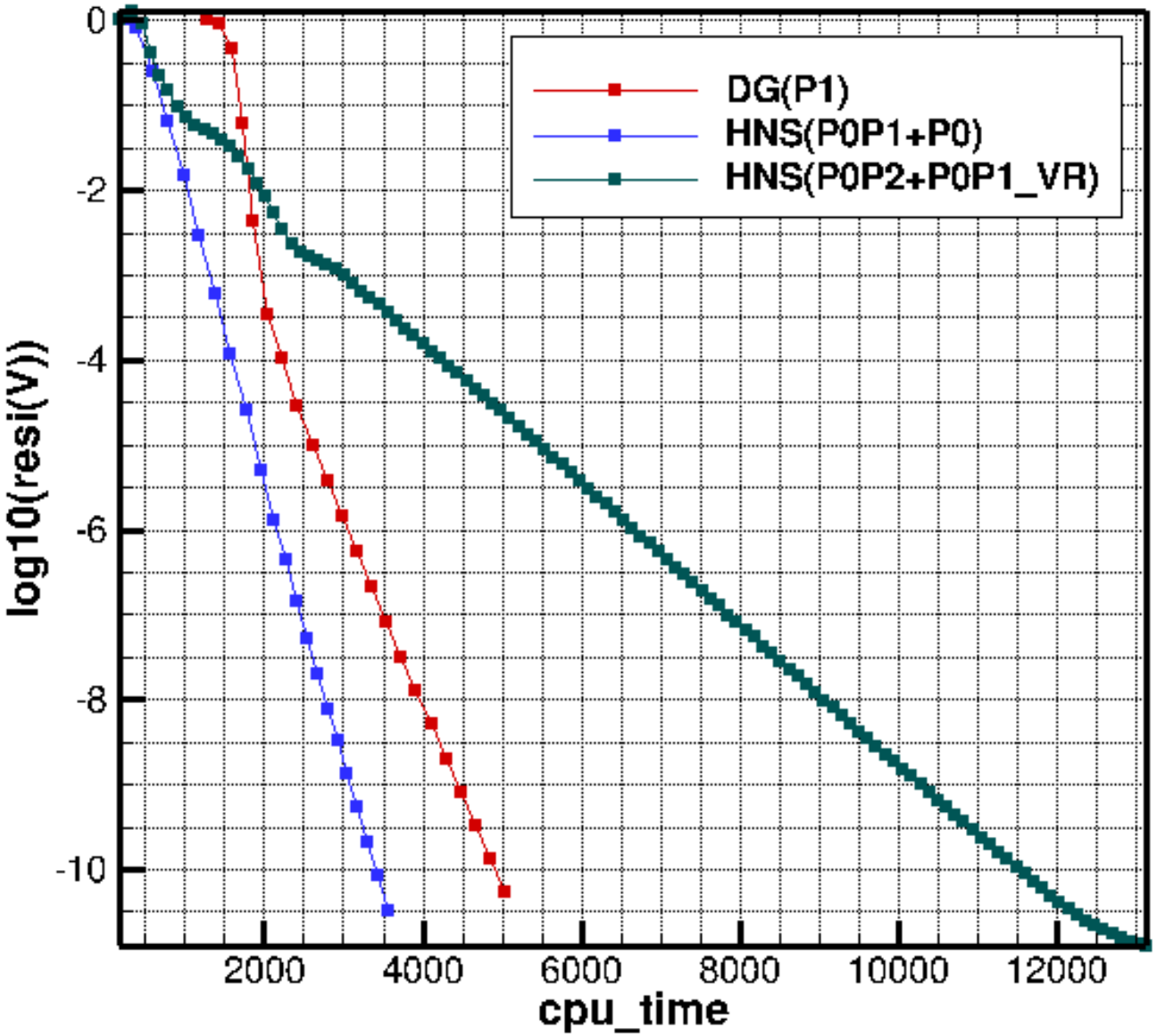
  37,274



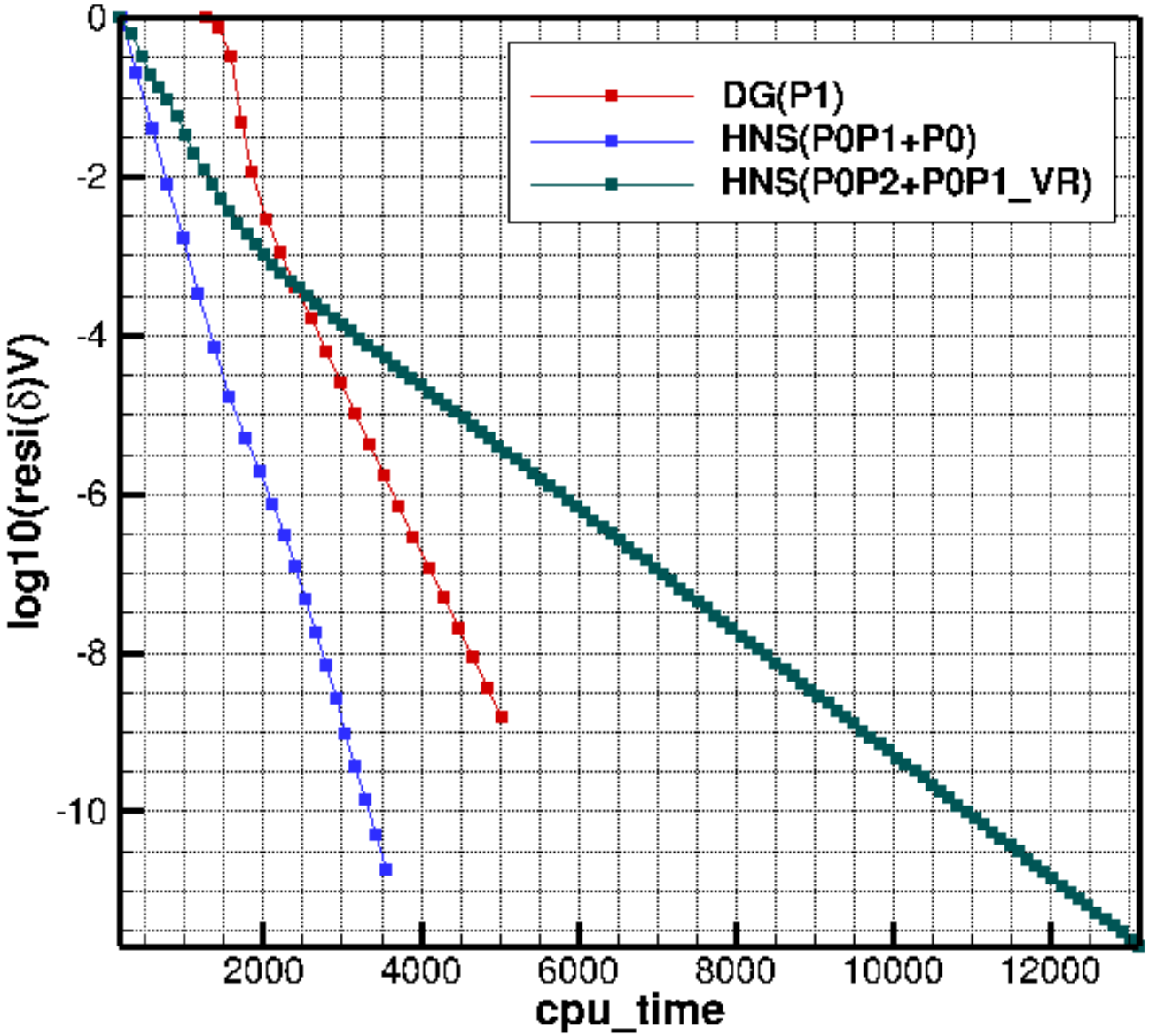
Compressible Viscous Flow (HNS(P_0P_{m+1}+P_0P_m)+rDG)

Same convergence criteria: one order of magnitude

For SGS solver.

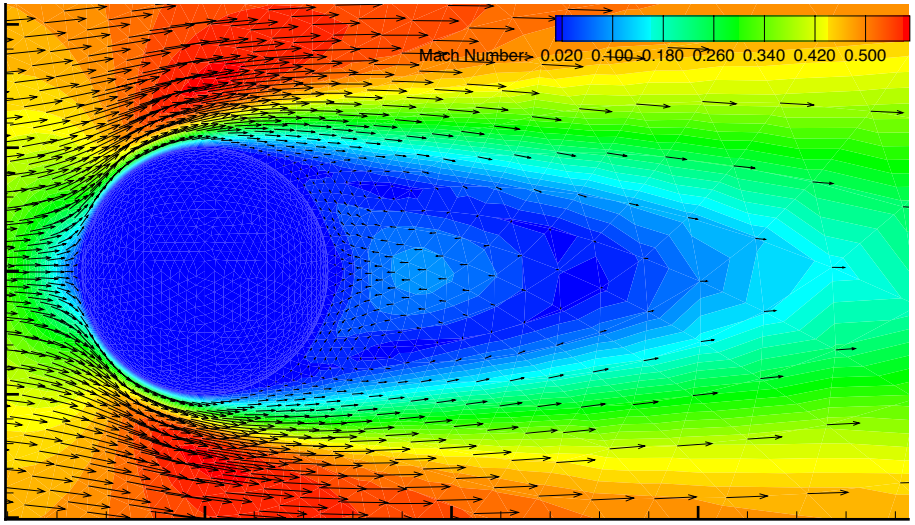


Compressible Viscous Flow (HNS(P_0P_{m+1}+P_0P_m)+rDG)

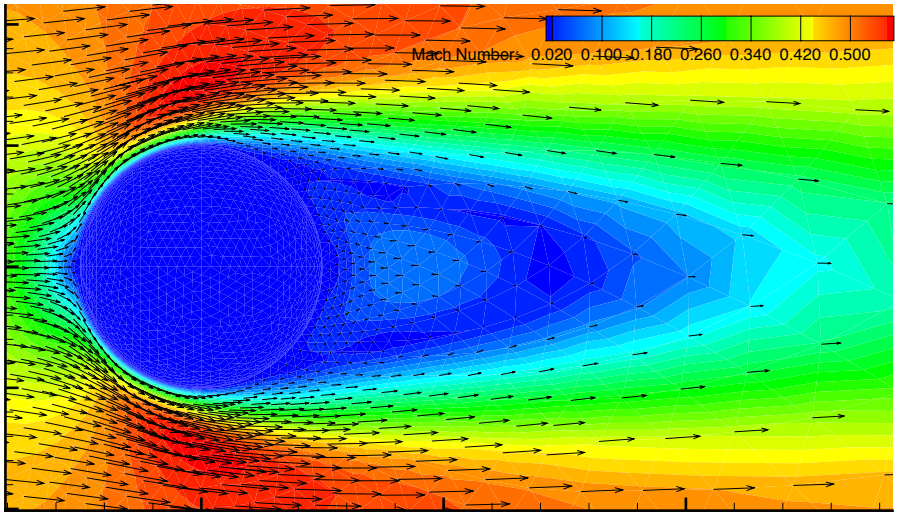


Compressible Viscous Flow (HNS(P_0P_{m+1}+P_0P_m)+rDG)

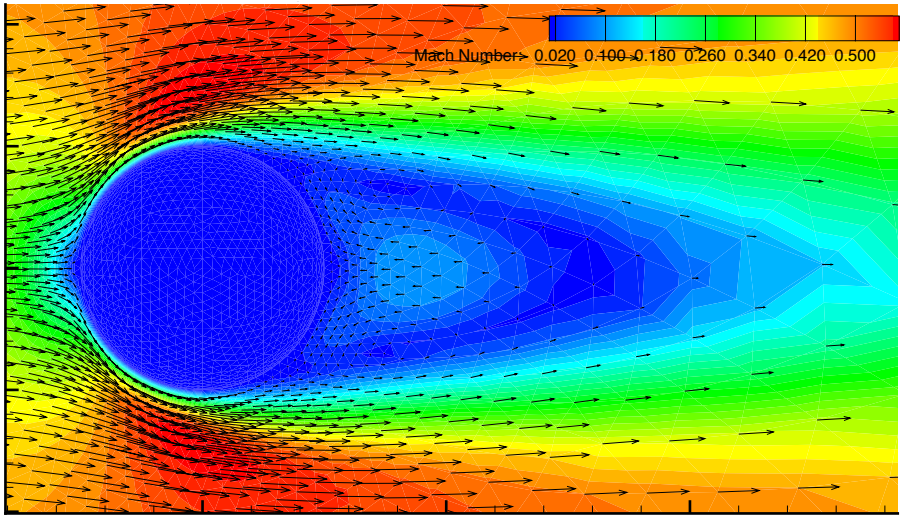
Mach Contours for Re=100



HNS(P0P1+P0)

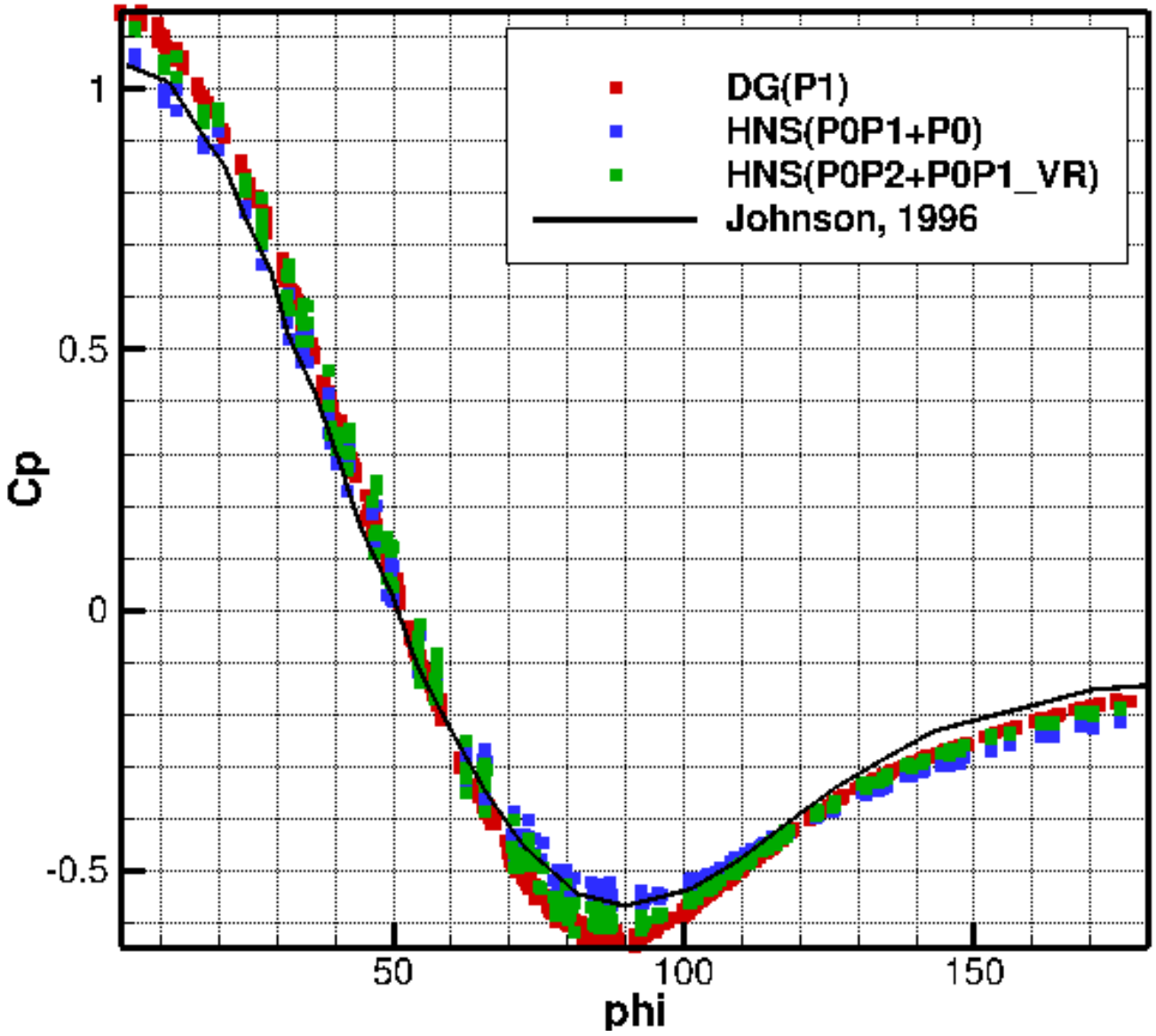


HNS(P0P2+P0P1_VR)



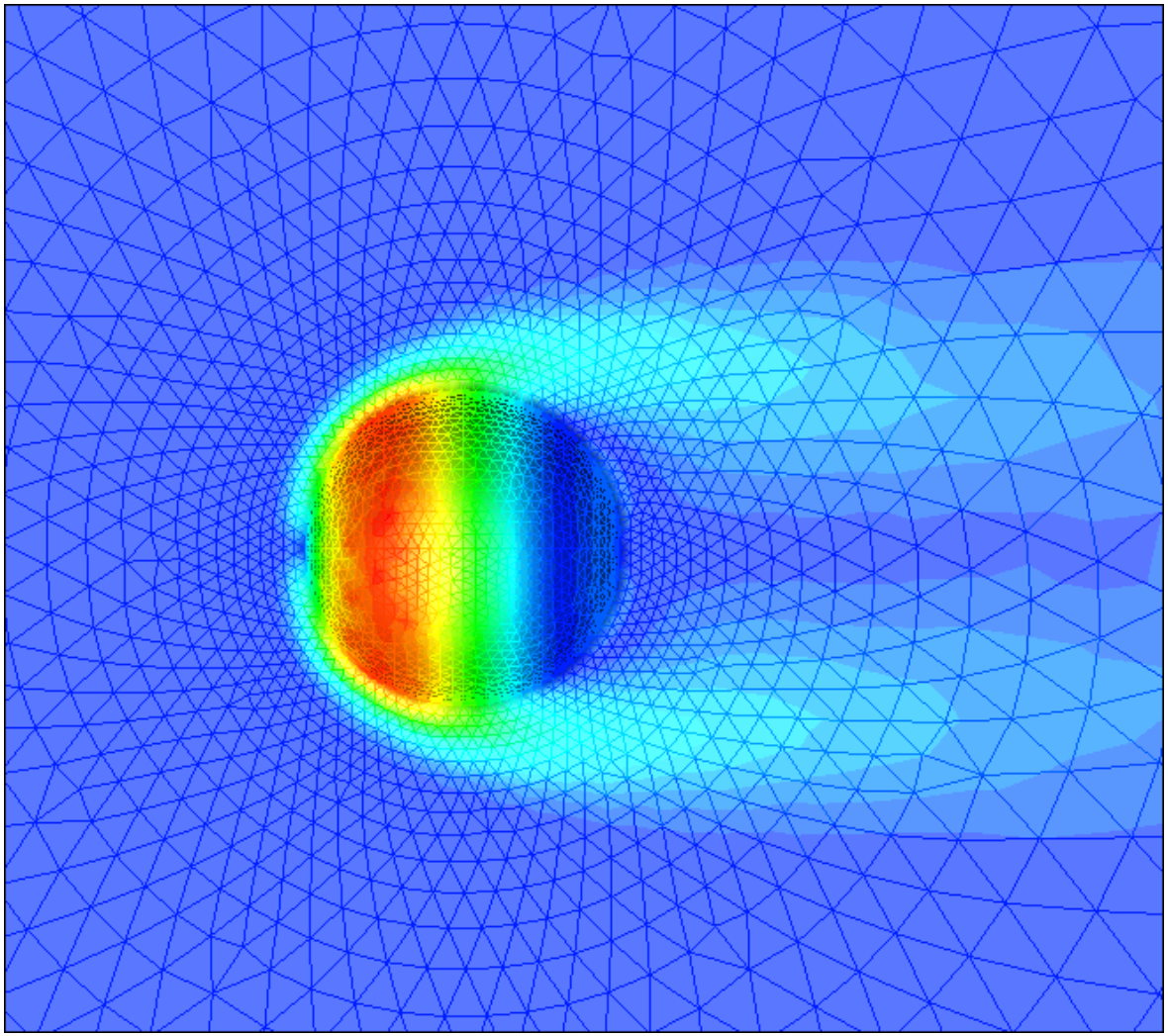
DG(P1)

Compressible Viscous Flow (HNS(P_0P_{m+1}+P_0P_m)+rDG)



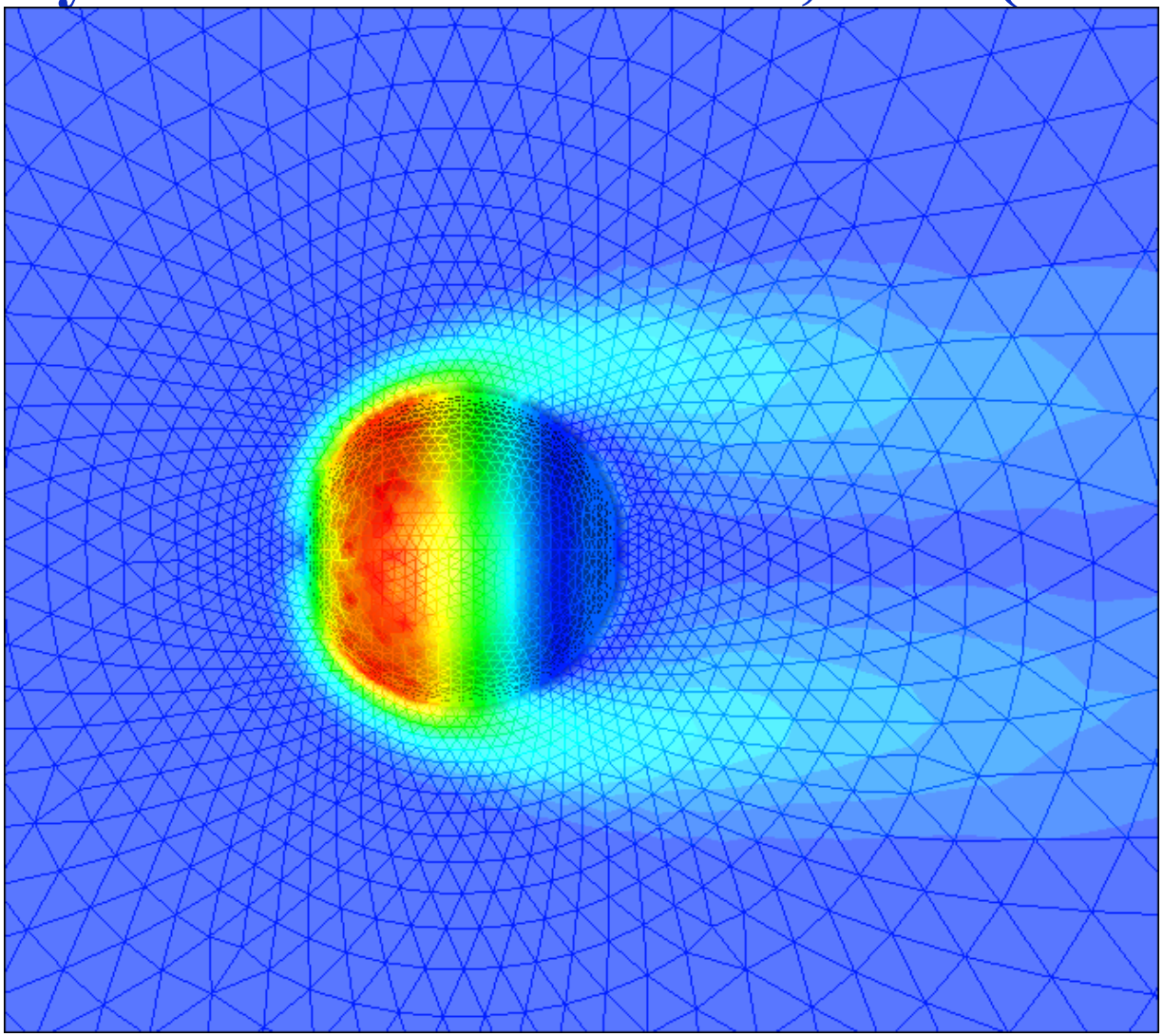
Compressible Viscous Flow (HNS(P_0P_{m+1}+P_0P_m)+rDG)

Vorticity Contours for Re=100, DG(P1)



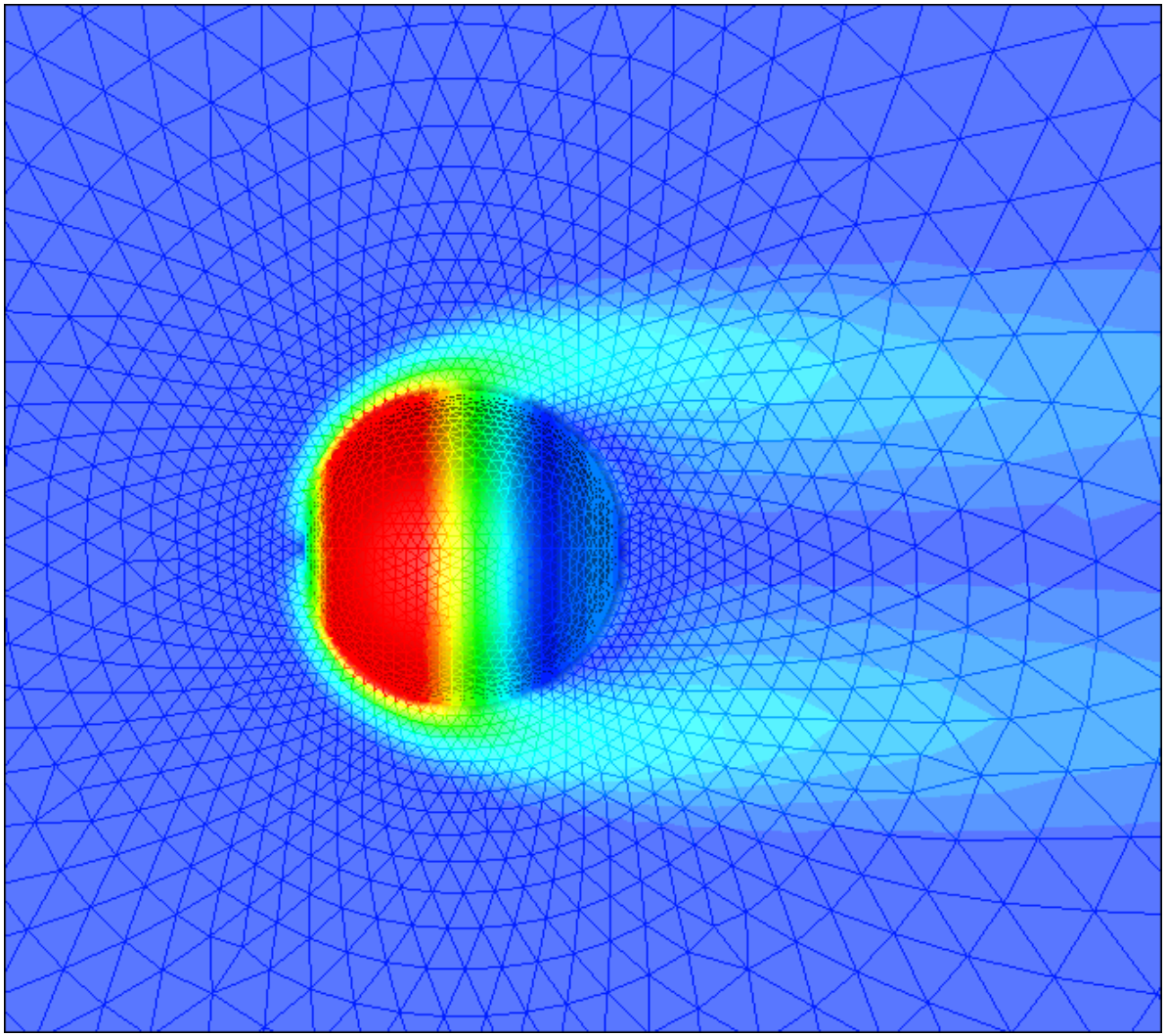
Compressible Viscous Flow (HNS(P_0P_{m+1}+P_0P_m)+rDG)

Vorticity Contours for Re=100, HNS(P0P1+P0)



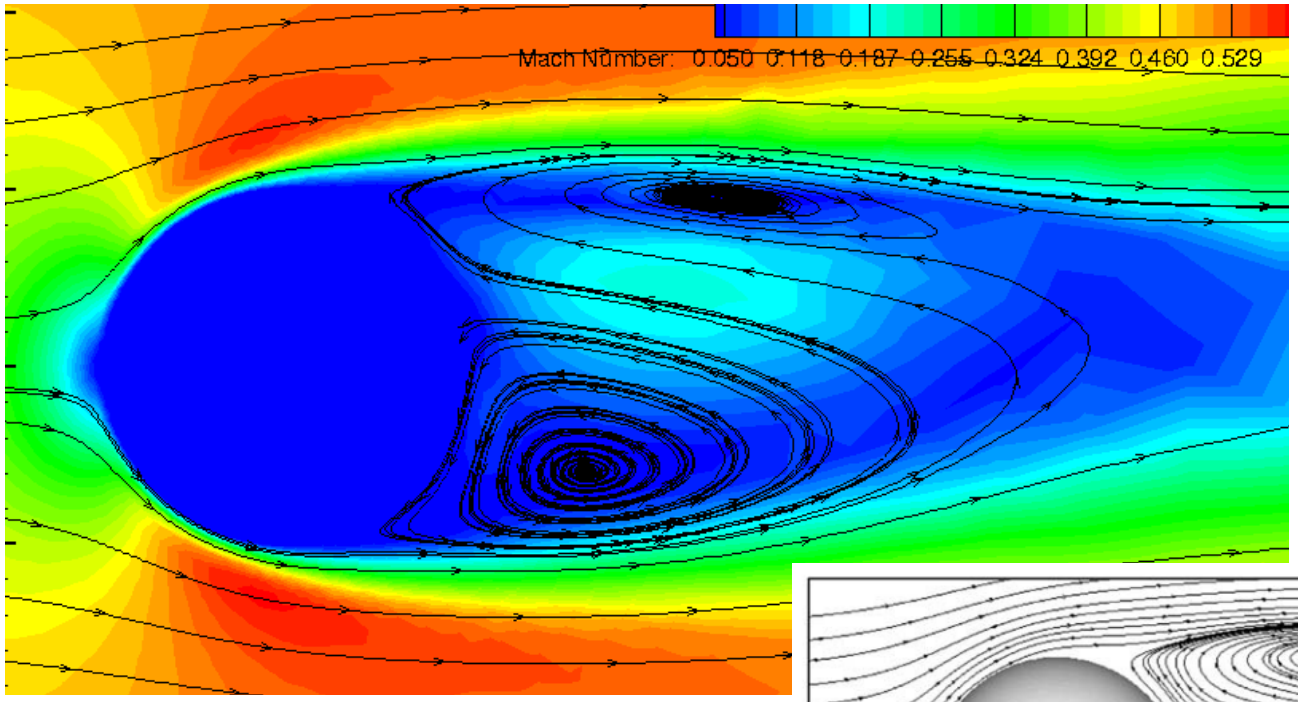
Compressible Viscous Flow (HNS(P_0P_{m+1}+P_0P_m)+rDG)

Vorticity Contours for Re=100, HNS(P0P2+P0P1_VR)

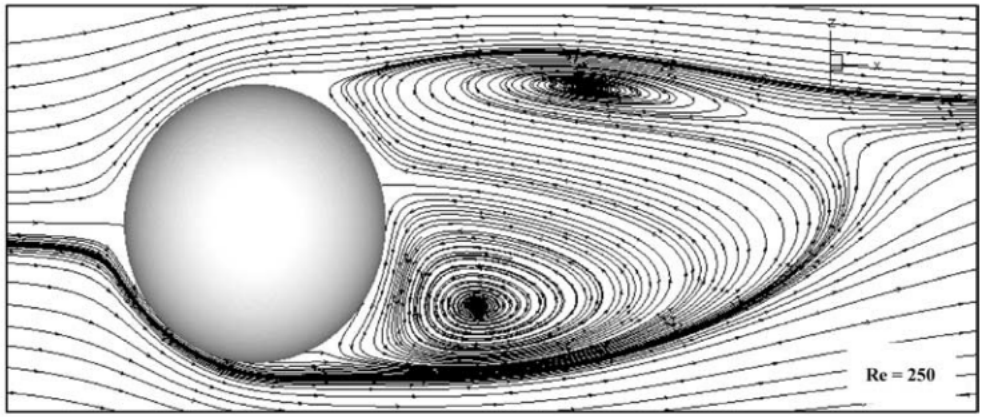


Compressible Viscous Flow (HNS(P_0P_{m+1}+P_0P_m)+rDG)

Mach Contours and Streamlines for Re=250



HNS(P0P2+P0P1(VR))

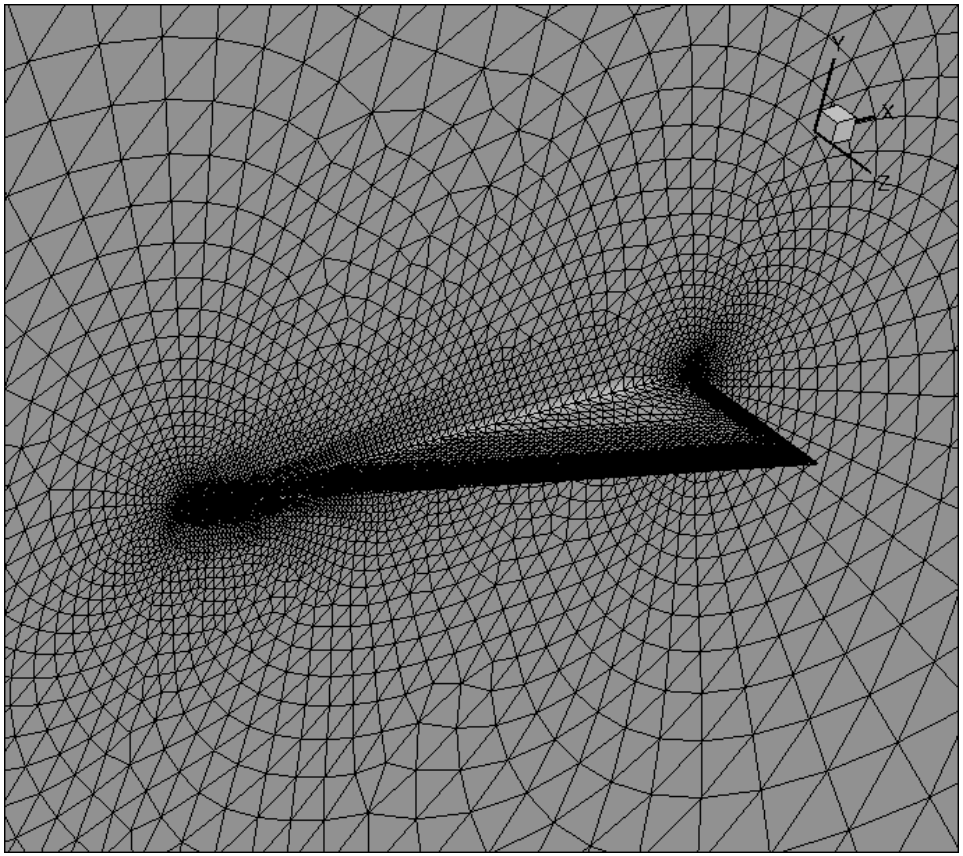


Gilmanov, 2003

Compressible Viscous Flow (HNS(P_0P_{m+1}+P_0P_m)+rDG)

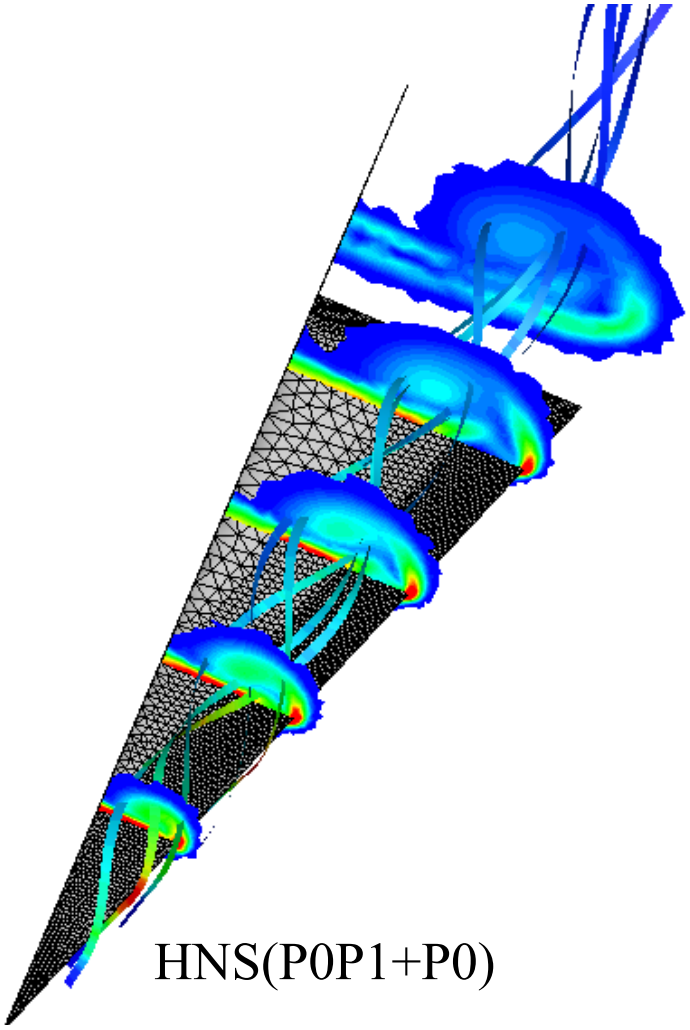
Case 9: Laminar Flow Past a Delta Wing (HNS+rDG)

- Mach = 0.3
- Re=4,000
- Angle of attack=12.5^o
- Number of tetrahedral: 670,680
- Number of points: 197,752

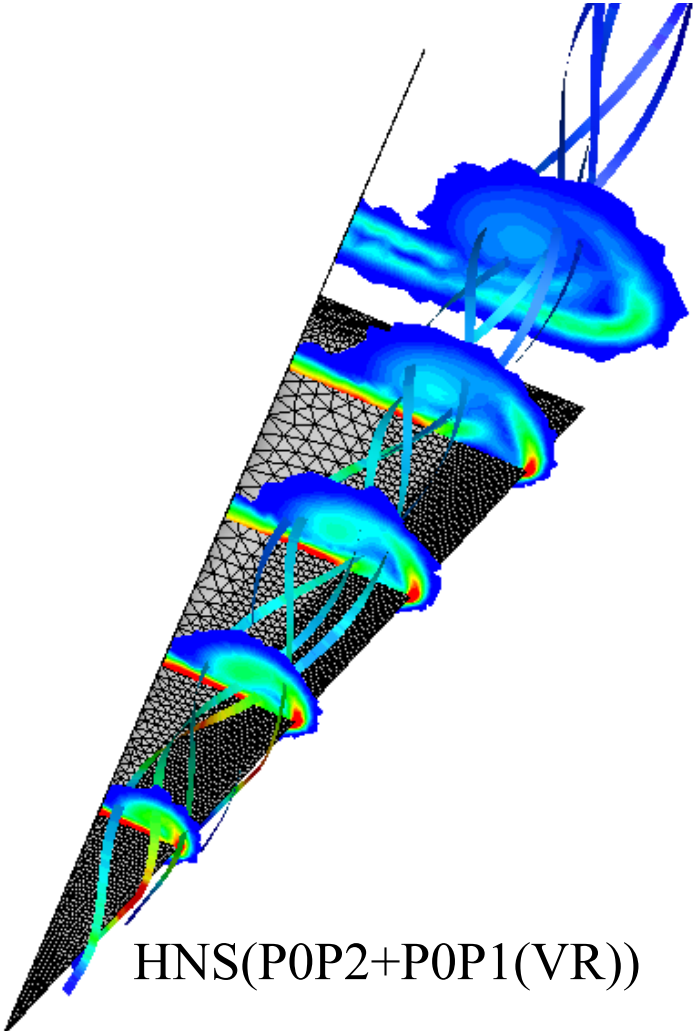


Compressible Viscous Flow (HNS(P_0P_{m+1}+P_0P_m)+rDG)

Vorticity Magnitude of Laminar Flow Past a Delta Wing



HNS(P0P1+P0)



HNS(P0P2+P0P1(VR))

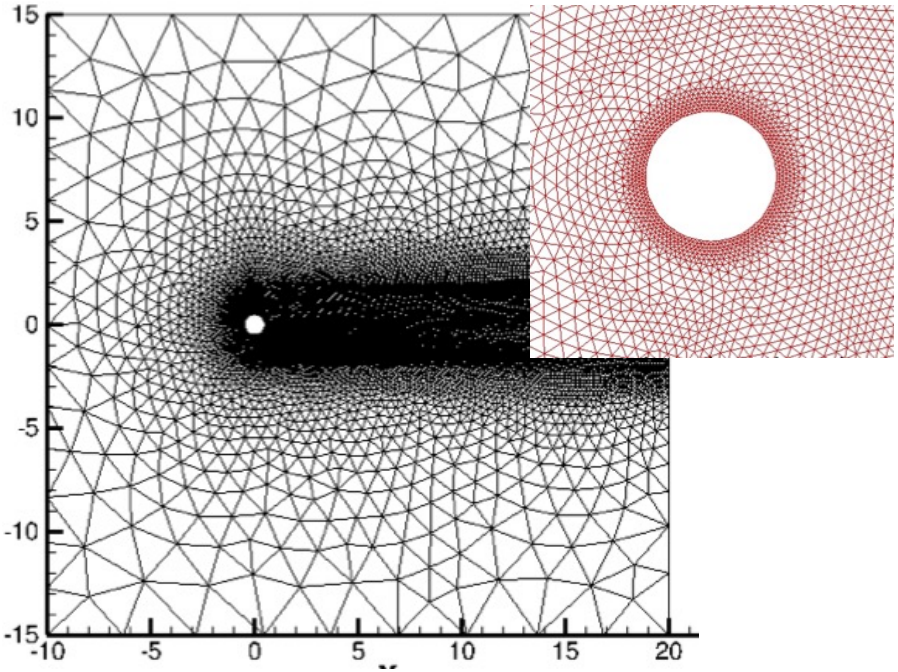
Compressible Viscous Flow (HNS(P_0P_{m+1}+P_0P_m)+rDG)

Case 10: Unsteady laminar flow past a circular cylinder

- Mach = 0.1
- Re=100
- Number of prismatic:

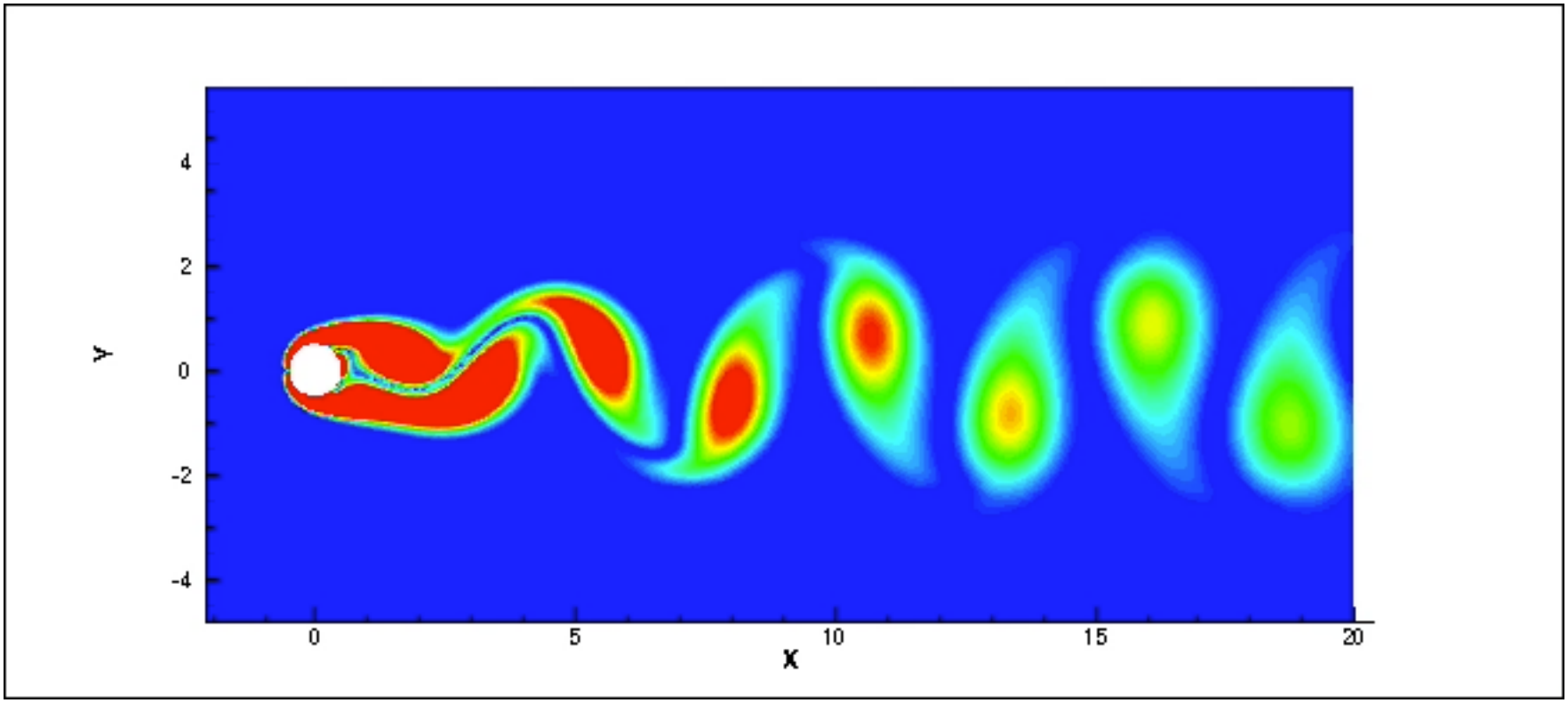
  21,809
- Number of points:

  22,804



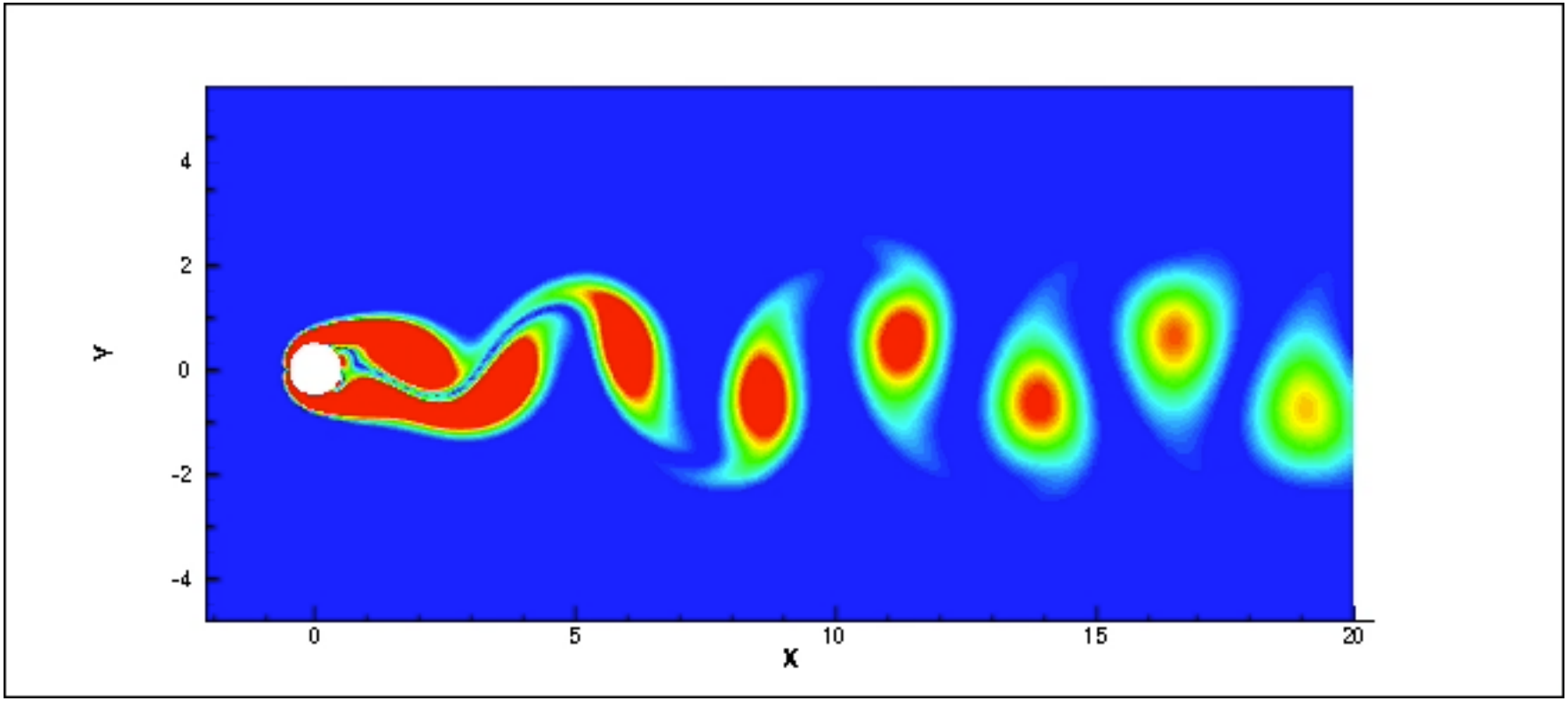
Compressible Viscous Flow (HNS(P_0P_2+P_0P_1_VR)+rDG)

Vorticity magnitude within half period, Dt=0.5

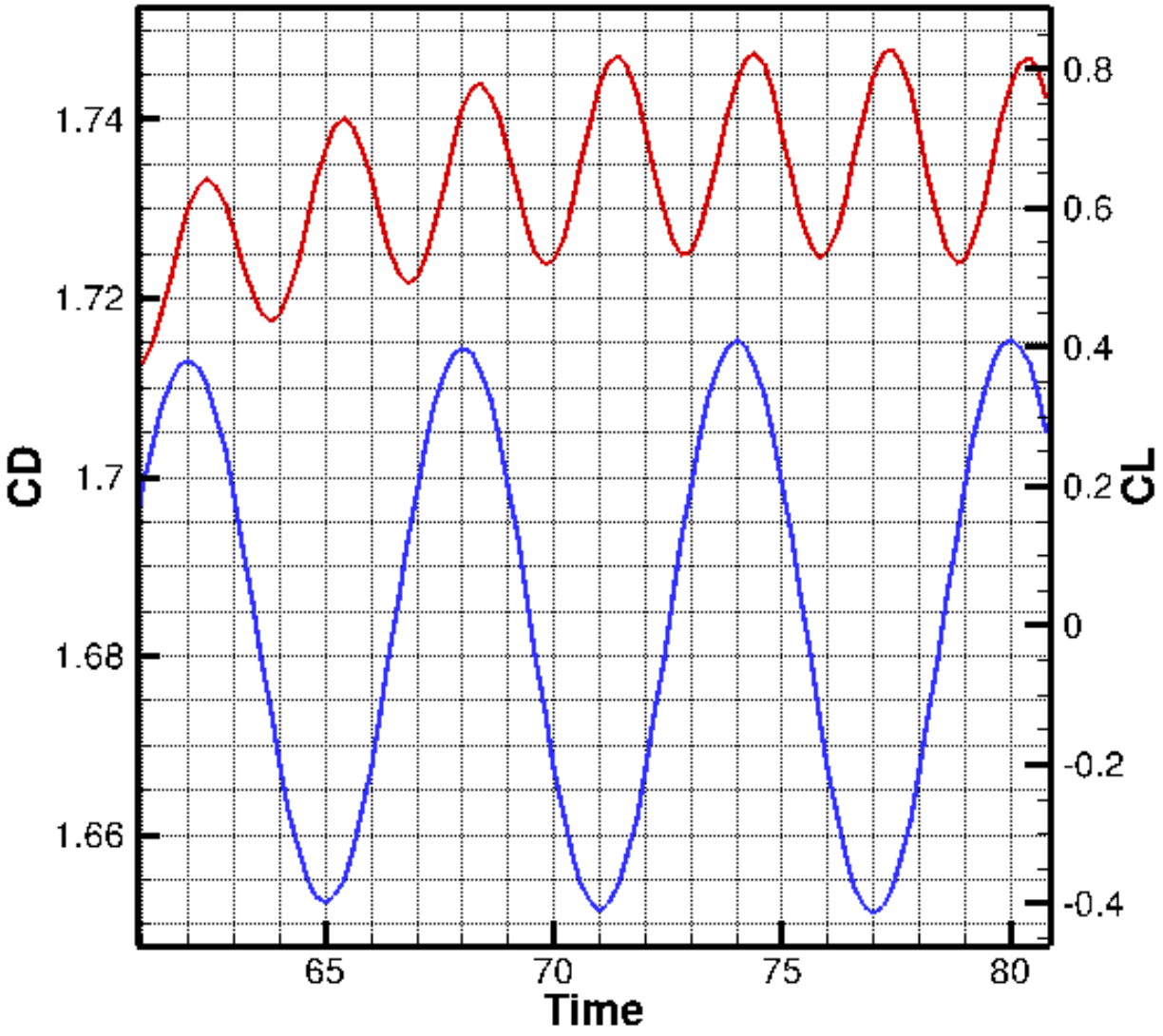


Compressible Viscous Flow (HNS(P_0P_2+P_0P_1_VR)+rDG)

Vorticity magnitude within half period, Dt=0.2



Compressible Viscous Flow (HNS(P_0P_2+P_0P_1_VR)+rDG)



Concluding Remarks

An rDG method with Variational Reconstruction has been implemented for Euler Equations

- K-exactness and designed order of convergence.
- Robustness.

HNS+rFV and HNS+rDG methods has been implemented for Navier-Stokes Equations

- Designed optimal order of accuracy for both primary variables and their gradients.
- HNS(P0P1+P0) converges faster than conventional DG(P1).
- HNS(P0P2+P0P1_VR) generates smoother vorticity than DG(P1).
- Feasibility to solve real flow problems.

Thank you! Questions?