Mingzhe Sun and Philip L. Roe, June 20, 2013 (Unpublished)

Viscous Active Flux method

1D heat conduction equation

For simplicity, let's consider active flux method for 1D heat conduction equation first. The heat conduction equation is,

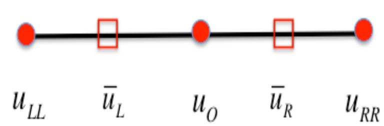
$$\frac{\partial u}{\partial t}=\kappa\frac{\partial^2 u}{\partial x^2} \tag{1}$$

where κ is the heat conduction coefficient. Now rewrite this formula into the conservative form as:

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}(-\kappa\frac{\partial u}{\partial x})=0 \tag{2}$$

In this way, the term $-\kappa u_x$ is regarded as the flux term and Finite Volume method could be applied to update the cell averages.

1.1. Reconstruction and flux term



Consider the stencil shown, where $\bar{u}_L,\bar{u}_R$ are cell averages and u_{LL},u_O,u_{RR} are point values. Now let's reconstruct u(x) separately in each cell as

$$u_L(x)=u_O+2(2u_O-3\bar{u}_L+u_{LL})\frac{x}{h}+3(u_O-2\bar{u}_L+u_{LL})(\frac{x}{h})^2 \tag{1.1.1}$$

$$u_R(x)=u_O-2(2u_O-3\bar{u}_R+u_{RR})\frac{x}{h}+3(u_O-2\bar{u}_R+u_{RR})(\frac{x}{h})^2 \tag{1.1.2}$$

The key feature of this reconstruction it that it keeps the conservative quality within each cell. Now let's find out the formula for flux term. Since $F=-\kappa u_x$, and from Taylor's expansion, we know

$$\begin{aligned}F(t)&=-\kappa[u_x(0)+tu_{xt}(0)+\frac{t^2}{2}u_{xtt}(0)+\cdots]\\&=-\kappa[u_x(0)+\kappa tu_{xxx}(0)+\frac{\kappa^2t^2}{2}u_{xxxxx}(0)+\cdots]\end{aligned} \tag{1.1.3}$$

In order to find a third-order scheme, the last term in (1.1.3) could be neglected here. There are so many ways to find the first and third derivatives of u with respect to x. We use Taylor's expansion to find the first derivative term. First, expand any value u with respect to u_O as

$$u = u_O + x\frac{\partial u}{\partial x} + \frac{x^2}{2!}\frac{\partial^2 u}{\partial x^2} + \frac{x^3}{3!}\frac{\partial^3 u}{\partial x^3} + \frac{x^4}{4!}\frac{\partial^4 u}{\partial x^4} + \cdots \tag{1.1.4}$$

So the point values are

$$u_{LL} = u_O - h\frac{\partial u}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 u}{\partial x^2} - \frac{h^3}{3!}\frac{\partial^3 u}{\partial x^3} + \frac{h^4}{4!}\frac{\partial^4 u}{\partial x^4} + \cdots \tag{1.1.5}$$

$$u_{RR} = u_O + h\frac{\partial u}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 u}{\partial x^2} + \frac{h^3}{3!}\frac{\partial^3 u}{\partial x^3} + \frac{h^4}{4!}\frac{\partial^4 u}{\partial x^4} + \cdots \tag{1.1.6}$$

The average values could be found as taking the integral then divide by the space interval h, so

$$\begin{aligned}\overline{u}_L &= \frac{1}{h}\int_{-h}^{0} u\,dx = \frac{1}{h}\int_{-h}^{0}\left(u_O + x\frac{\partial u}{\partial x} + \frac{x^2}{2!}\frac{\partial^2 u}{\partial x^2} + \frac{x^3}{3!}\frac{\partial^3 u}{\partial x^3} + \frac{x^4}{4!}\frac{\partial^4 u}{\partial x^4} + \cdots\right)dx \\ &= u_O - \frac{h}{2!}\frac{\partial u}{\partial x} + \frac{h^2}{3!}\frac{\partial^2 u}{\partial x^2} - \frac{h^3}{4!}\frac{\partial^3 u}{\partial x^3} + \frac{h^4}{5!}\frac{\partial^4 u}{\partial x^4} - \cdots\end{aligned} \tag{1.1.7}$$

$$\begin{aligned}\overline{u}_R &= \frac{1}{h}\int_{0}^{h} u\,dx = \frac{1}{h}\int_{0}^{h}\left(u_O + x\frac{\partial u}{\partial x} + \frac{x^2}{2!}\frac{\partial^2 u}{\partial x^2} + \frac{x^3}{3!}\frac{\partial^3 u}{\partial x^3} + \frac{x^4}{4!}\frac{\partial^4 u}{\partial x^4} + \cdots\right)dx \\ &= u_O + \frac{h}{2!}\frac{\partial u}{\partial x} + \frac{h^2}{3!}\frac{\partial^2 u}{\partial x^2} + \frac{h^3}{4!}\frac{\partial^3 u}{\partial x^3} + \frac{h^4}{5!}\frac{\partial^4 u}{\partial x^4} + \cdots\end{aligned} \tag{1.1.8}$$

Firstly, let's find the first term ($-\kappa u_x(0)$) of the flux. Giving different coefficients to these formulas and let the second, third and forth derivative terms cancel out, we could get

$$\frac{\partial u}{\partial x} = \frac{4(\overline{u}_R - \overline{u}_L) - (u_{RR} - u_{LL})}{2h} + O(h^4) \tag{1.1.9}$$

To the part of finding $-\kappa^2 t u_{xxx}(0)$, use the reconstruction polynomial functions and take the second derivative terms of each part, then we get

$$u_{Lxx} = \frac{6}{h^2}(u_O - 2\overline{u}_L + u_{LL}) \tag{1.1.10}$$

$$u_{Rxx} = \frac{6}{h^2}(u_O - 2\overline{u}_R + u_{RR}) \tag{1.1.11}$$

In this way, the third derivatives could be expressed as

$$\frac{\partial^3 u}{\partial x^3} = \frac{u_{Rxx} - u_{Lxx}}{h} = \frac{6}{h^3}(2\overline{u}_L - 2\overline{u}_R + u_{RR} - u_{LL}) + O(h^2) \tag{1.1.12}$$

Now let's check out the order of the scheme. Since

$$F(t) = -\kappa[u_x(0) + \kappa t u_{xxx}(0) + \cdots] \qquad (1.1.13)$$

Then

$$O(F) = O(u_x(0)) + O(tu_{xxx}(0)) \qquad (1.1.14)$$

Time is chosen proportionally to h^2, so

$$O(F) = O(h^4) + O(th^2) = O(h^4) \qquad (1.1.15)$$

Forward Euler method in time is used here and then the order of the discretization scheme turns out to be:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = O(t) + \frac{O(h^4)}{O(h)} + \cdots = O(t) + O(h^3) + \cdots \qquad (1.1.16)$$

So the scheme is first order in time and third order in space.

1.2. Point value updating

We use a different strategy to update the point value. From Taylor's expansion, we have

$$u(t) = u(0) + t\frac{\partial u}{\partial t} + \frac{t^2}{2!}\frac{\partial^2 u}{\partial t^2} + \frac{t^3}{3!}\frac{\partial^3 u}{\partial t^3} + \cdots \qquad (1.2.1)$$

Recall the heat conduction equation

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \qquad (1.2.2)$$

So the second derivative of u with respect to t is

$$\frac{\partial^2 u}{\partial t^2} = \kappa^2 \frac{\partial^4 u}{\partial x^4} \qquad (1.2.3)$$

Now the time derivatives in (1.2.1) are related to spatial derivatives so that the strategy we used in finding the flux term could also be used here and finally we could get

$$\frac{\partial u}{\partial t} = \kappa \frac{-3(u_{RR} + u_{LL}) - 24u_o + 15(\overline{u}_L + \overline{u}_R)}{2h^2} + O(h^4) \qquad (1.2.4)$$

$$\frac{\partial^2 u}{\partial t^2} = \kappa^2 \frac{30(u_{RR} + u_{LL} + 4u_o - 3\overline{u}_L - 3\overline{u}_R)}{h^4} + O(h^2) \qquad (1.2.5)$$

Substitute formula (1.2.4) and (1.2.5) in (1.2.1) we could update the value of the point, of which the order is $O(h^4)$.

1.3. Stability analysis

From Finite Volume Method, the cell averages are updated conservatively from the flux functions

$$\overline{u}_j^{n+1} = \overline{u}_j^n - \frac{\Delta t}{h}(\overline{f}_{j+\frac{1}{2}} - \overline{f}_{j-\frac{1}{2}}) \tag{1.3.1}$$

The formation of flux term on the interface at the right side of a certain cell is

$$\begin{aligned} f_{j+\frac{1}{2}}(t) &= -\kappa[u_x(0)\Big|_{x=x_{j+1/2}} + \kappa t u_{xxx}(0)\Big|_{x=x_{j+1/2}}] \\ &= -\kappa\{\frac{1}{h}[2(\overline{u}_{j+1}^n - \overline{u}_j^n) - \frac{1}{2}(u^n_{j+\frac{3}{2}} - u^n_{j-\frac{1}{2}})] + \kappa t[\frac{12}{h^3}(\overline{u}_j^n - \overline{u}_{j+1}^n) + \frac{6}{h^3}(u^n_{j+\frac{3}{2}} - u^n_{j-\frac{1}{2}})]\} \\ &= -\frac{\kappa}{h}[(2 - \frac{12\kappa t}{h^2})(\overline{u}_{j+1}^n - \overline{u}_j^n) - (\frac{1}{2} - \frac{6\kappa t}{h^2})(u^n_{j+\frac{3}{2}} - u^n_{j-\frac{1}{2}})] \end{aligned} \tag{1.3.2}$$

From Simpson's law, we take the average flux on the right interface as

$$\begin{aligned} \overline{f}_{j+\frac{1}{2}} &= \frac{1}{6}[f_{j+\frac{1}{2}}(0) + 4f_{j+\frac{1}{2}}(\frac{\Delta t}{2}) + f_{j+\frac{1}{2}}(\Delta t)] \\ &= -\frac{\kappa}{h}[2(1 - \frac{3\kappa\Delta t}{h^2})(\overline{u}_{j+1}^n - \overline{u}_j^n) - (\frac{1}{2} - \frac{3\kappa\Delta t}{h^2})(u^n_{j+\frac{3}{2}} - u^n_{j-\frac{1}{2}})] \end{aligned} \tag{1.3.3}$$

Let $\mu = \frac{\kappa\Delta t}{h^2}$, then formula (1.3.3) yields to

$$\overline{f}_{j+\frac{1}{2}} = -\frac{\kappa}{h}[2(1-3\mu)(\overline{u}_{j+1}^n - \overline{u}_j^n) - (\frac{1}{2} - 3\mu)(u^n_{j+\frac{3}{2}} - u^n_{j-\frac{1}{2}})] \tag{1.3.4}$$

By changing the subscript in formula (1.3.4), we could easily get the average flux term on the left interface as

$$\overline{f}_{j-\frac{1}{2}} = -\frac{\kappa}{h}[2(1-3\mu)(\overline{u}_j^n - \overline{u}_{j-1}^n) - (\frac{1}{2} - 3\mu)(u^n_{j+\frac{1}{2}} - u^n_{j-\frac{3}{2}})] \tag{1.3.5}$$

Subtract formula (1.3.5) from (1.3.4), we get

$$\overline{f}_{j+\frac{1}{2}} - \overline{f}_{j-\frac{1}{2}} = -\frac{\kappa}{h}[2(1-3\mu)(\overline{u}_{j+1}^n - 2\overline{u}_j^n + \overline{u}_{j-1}^n) - (\frac{1}{2} - 3\mu)(u^n_{j+\frac{3}{2}} - u^n_{j+\frac{1}{2}} - u^n_{j-\frac{1}{2}} + u^n_{j-\frac{3}{2}})] \tag{1.3.6}$$

Substitute (1.3.6) into (1.3.1), we have

$$\begin{aligned} \overline{u}_j^{n+1} &= \overline{u}_j^n - \frac{\Delta t}{h}(\overline{f}_{j+\frac{1}{2}} - \overline{f}_{j-\frac{1}{2}}) \\ &= \overline{u}_j^n + [2\mu(1-3\mu)(\overline{u}_{j+1}^n - 2\overline{u}_j^n + \overline{u}_{j-1}^n) - \mu(\frac{1}{2} - 3\mu)(u^n_{j+\frac{3}{2}} - u^n_{j+\frac{1}{2}} - u^n_{j-\frac{1}{2}} + u^n_{j-\frac{3}{2}})] \\ &= [1 - 4\mu(1-3\mu)]\overline{u}_j^n + 2\mu(1-3\mu)(\overline{u}_{j+1}^n + \overline{u}_{j-1}^n) - \mu(\frac{1}{2} - 3\mu)(u^n_{j+\frac{3}{2}} - u^n_{j+\frac{1}{2}} - u^n_{j-\frac{1}{2}} + u^n_{j-\frac{3}{2}})] \end{aligned} \tag{1.3.7}$$

For the point value updating process, recall from formula (1.2.4) and (1.2.5), We have

$$u_{j+\frac{1}{2}}^{n+1}=u_{j+\frac{1}{2}}^{n}+\frac{\kappa\Delta t}{2h^2}(-3(u_{j+\frac{3}{2}}^{n}+u_{j-\frac{1}{2}}^{n})-24u_{j+\frac{1}{2}}^{n}+15(\overline{u}_{j+1}^{n}+\overline{u}_{j}^{n}))$$
$$+15(\frac{\kappa\Delta t}{h^2})^2(u_{j+\frac{3}{2}}^{n}+u_{j-\frac{1}{2}}^{n}+4u_{j+\frac{1}{2}}^{n}-3(\overline{u}_{j+1}^{n}+\overline{u}_{j}^{n})) \tag{1.3.8}$$
$$=(1-12\mu+60\mu^2)u_{j+\frac{1}{2}}^{n}+(15\mu^2-\frac{3}{2}\mu)(u_{j+\frac{3}{2}}^{n}+u_{j-\frac{1}{2}}^{n})+(\frac{15}{2}\mu-45\mu^2)(\overline{u}_{j+1}^{n}+\overline{u}_{j}^{n})$$

Now let's find the amplification factor of the method from a Fourier analysis. This time it should be a 2x2 matrix.
For the average value

$$\overline{u}_{j}^{n+1}=[1-4\mu(1-3\mu)]\overline{u}_{j}^{n}+2\mu(1-3\mu)(\overline{u}_{j+1}^{n}+\overline{u}_{j-1}^{n})-\mu(\frac{1}{2}-3\mu)(u_{j+\frac{3}{2}}^{n}-u_{j+\frac{1}{2}}^{n}-u_{j-\frac{1}{2}}^{n}+u_{j-\frac{3}{2}}^{n})$$
$$=[1-4\mu(1-3\mu)+2\mu(1-3\mu)(e^{i\phi}+e^{-i\phi})]\overline{u}_{j}^{n}-\mu(\frac{1}{2}-3\mu)(e^{i\phi}-1-e^{-i\phi}+e^{-2i\phi})u_{j+\frac{1}{2}}^{n}$$
$$=[1-4\mu(1-3\mu)(1-\cos\phi)]\overline{u}_{j}^{n}-\mu(\frac{1}{2}-3\mu)(e^{-2i\phi}-1)(1-e^{i\phi})u_{j+\frac{1}{2}}^{n} \tag{1.3.9}$$

For the point value

$$u_{j+\frac{1}{2}}^{n+1}=(1-12\mu+60\mu^2)u_{j+\frac{1}{2}}^{n}+(15\mu^2-\frac{3}{2}\mu)(u_{j+\frac{3}{2}}^{n}+u_{j-\frac{1}{2}}^{n})+(\frac{15}{2}\mu-45\mu^2)(\overline{u}_{j+1}^{n}+\overline{u}_{j}^{n})$$
$$=(1-12\mu+60\mu^2+(15\mu^2-\frac{3}{2}\mu)(e^{i\phi}+e^{-i\phi}))u_{j+\frac{1}{2}}^{n}+(\frac{15}{2}\mu-45\mu^2)(1+e^{i\phi})\overline{u}_{j}^{n}$$
$$=(1-12\mu+60\mu^2+(30\mu^2-3\mu)\cos\phi)u_{j+\frac{1}{2}}^{n}+(\frac{15}{2}\mu-45\mu^2)(1+e^{i\phi})\overline{u}_{j}^{n} \tag{1.3.10}$$

Rewrite the amplification factor matrix in the form,

$$\begin{pmatrix}\overline{u}_j\\ u_{j+\frac{1}{2}}\end{pmatrix}^{n+1}=G\begin{pmatrix}\overline{u}_j\\ u_{j+\frac{1}{2}}\end{pmatrix}^{n}=\begin{pmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix}\begin{pmatrix}\overline{u}_j\\ u_{j+\frac{1}{2}}\end{pmatrix}^{n} \tag{1.3.11}$$

Where

$$\begin{cases}g_{11}=1-4\mu(1-3\mu)(1-\cos\phi)\\ g_{12}=-\mu(\frac{1}{2}-3\mu)(e^{-2i\phi}-1)(1-e^{i\phi})\\ g_{21}=(\frac{15}{2}\mu-45\mu^2)(1+e^{i\phi})\\ g_{22}=1-12\mu+60\mu^2+(30\mu^2-3\mu)\cos\phi\end{cases} \tag{1.3.12}$$

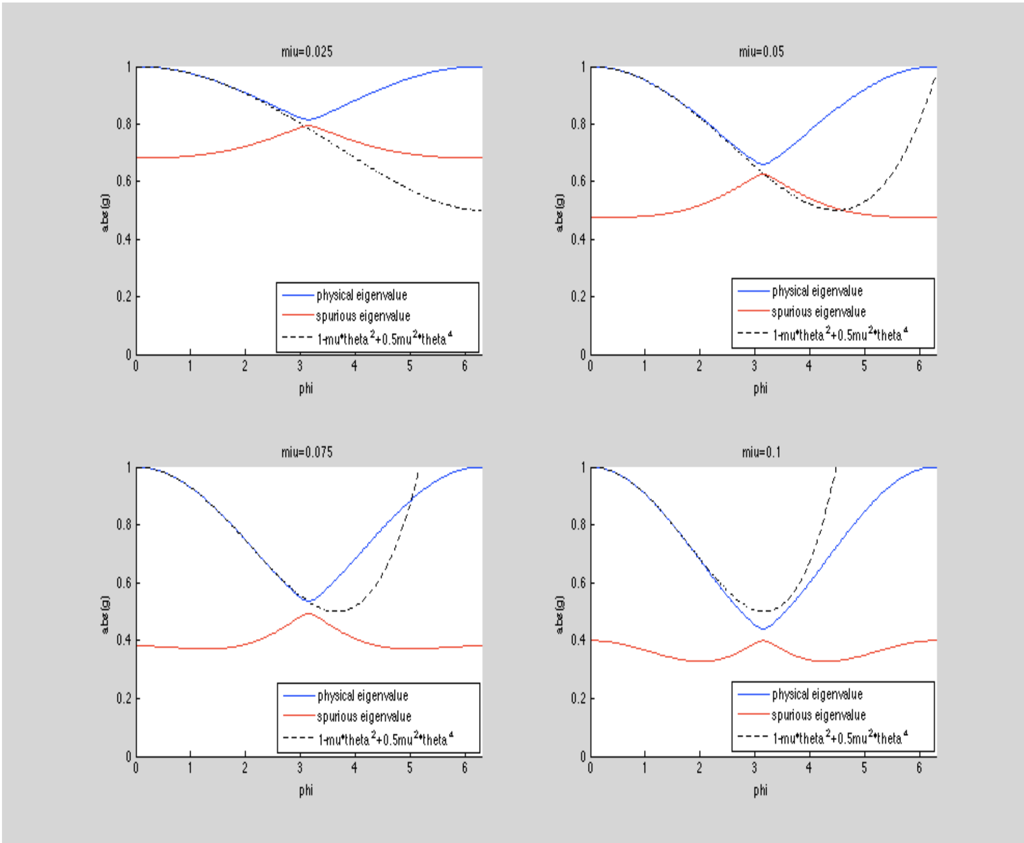


Fig. 1.3.1 Eigenvalues of amplification matrix

In Fig. 1.3.1, four plots of eigenvalues of the amplification factor matrix with respect to phase angle ϕ are shown with different choices of μ. The blue line shows the physical eigenvalue and the black dashed line is the approximated solution, from the similarity degree we could approximately know the order of the scheme. The red line is the spurious mode.

Now let's exactly analyze the order of the scheme. The way to do so is to find Taylor's expansion of the physical eigenvalue with respect to $\phi = 0$, i.e.

$$\lambda = \lambda\Big|_{\phi=0} + \frac{\partial \lambda}{\partial \phi}\Big|_{\phi=0}\phi + \frac{\partial^2 \lambda}{\partial \phi^2}\Big|_{\phi=0}\frac{\phi^2}{2!} + \frac{\partial^3 \lambda}{\partial \phi^3}\Big|_{\phi=0}\frac{\phi^3}{3!} + \ldots \tag{1.3.13}$$

And the eigenvalues could be calculated as

$$\lambda_{1,2} = \frac{g_{11} + g_{22} \pm \sqrt{(g_{11} + g_{22})^2 - 4(g_{11}g_{22} - g_{12}g_{21})}}{2} \tag{1.3.14}$$

After some arrangement we could finally find,

$$\lambda_{physical} = 1 - \mu\phi^2 + \frac{\mu^2}{2}\phi^4 + \frac{-12\mu^2 + \mu}{360}\phi^6 + O[\phi]^7 \tag{1.3.15}$$

The physical eigenvalue should approximate $e^{-\mu\theta^2}$, of which the Taylor expansion is

$$e^{-\mu\theta^2} = 1 - \mu\theta^2 + \frac{\mu^2}{2}\theta^4 - \frac{\mu^3}{6}\theta^6 + O[\theta]^7 \tag{1.3.16}$$

So we see the scheme is of 4th order but because of the influence from spurious mode it's actually less than 4th order.

1.4. Test case: 1D heat conduction

The initial condition is chosen at time t=0.1s of the temperature distribution

$$u(x,t) = \frac{1}{\sqrt{4\kappa t}}\exp\left(-\frac{x^2}{4\kappa t}\right) \tag{1.4.1}$$

The solution time is chosen at time t=0.5s so the exact solution is easily known. Time step is chosen as $dt = \frac{\mu}{\kappa}h^2$, where $\mu = 0.1$, $\kappa = 0.1$.

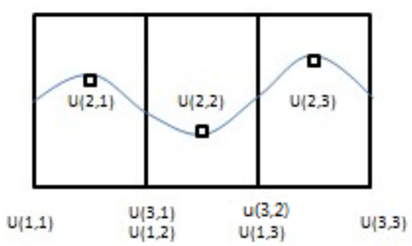
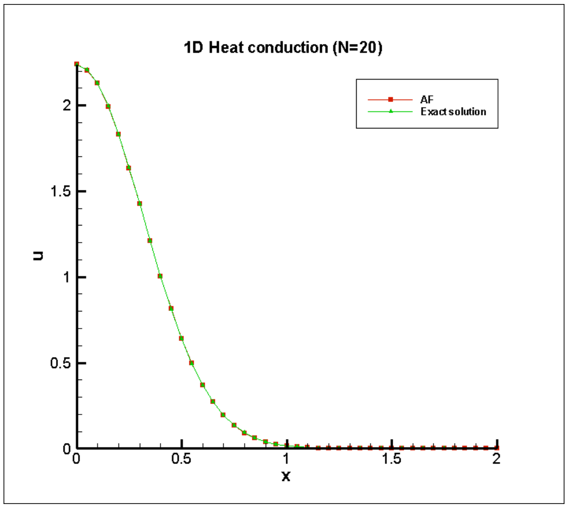


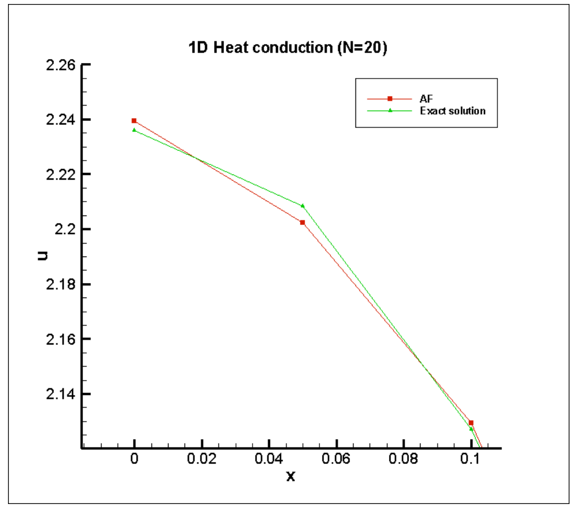
Fig. 1.4.1 Stencil on the left boundary

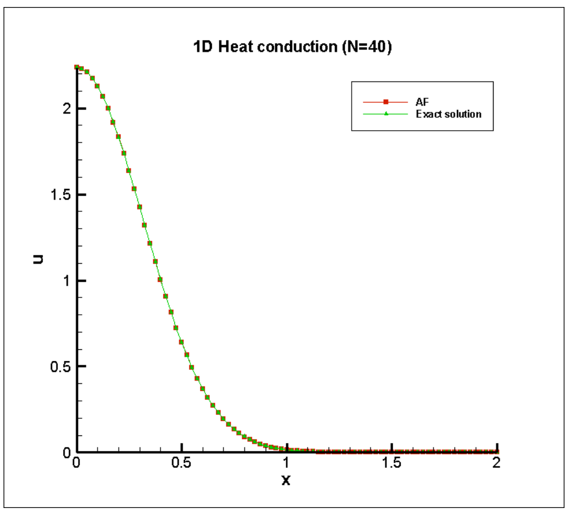
Take the boundary condition far away as $\frac{\partial u}{\partial x} = 0$. On the left boundary, introduce ghost cells and ghost interfaces to insist that the cell values are symmetric and the interface values antisymmetric, i.e. $u(1,1) = u(3,2), u(2,1) = u(2,2)$. For the value of $u(3,1)$, we aim to keep the curvature of the reconstruction polynomial formula, so $u(3,1) = (4u(3,2) - u(2,2))/3$.

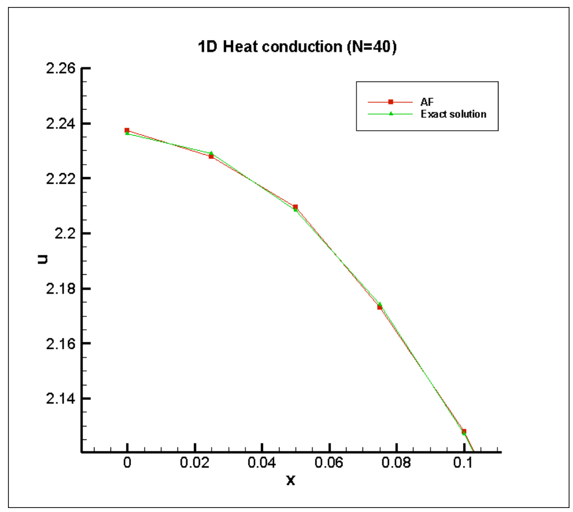
Results

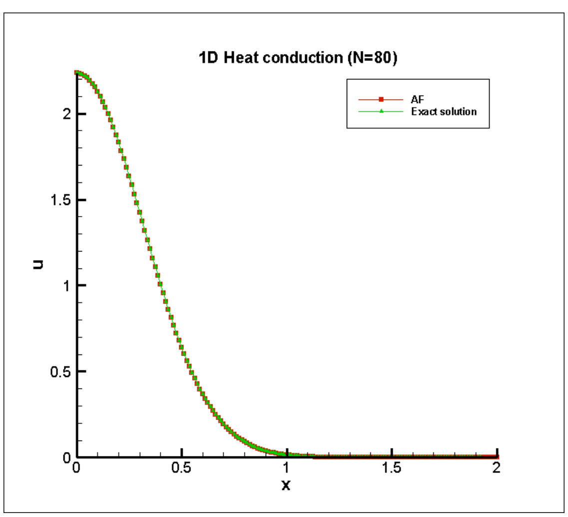
Solutions of cell averages with different grid numbers (N=20, 40, 80) are shown here. The figures on the right side show the details of the left boundary area. The AF method gives quite good accuracy that the solution is approximately to be the exact one.

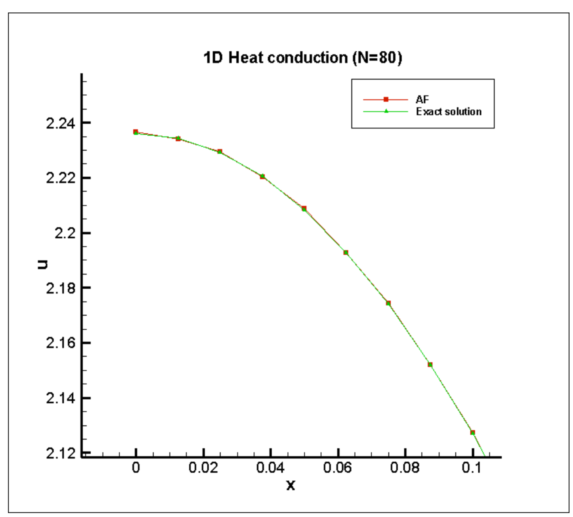

1D Heat conduction (N=20)
AF
Exact solution
u
x


1D Heat conduction (N=20)
AF
Exact solution
u
x


1D Heat conduction (N=40)
AF
Exact solution
u
x


1D Heat conduction (N=40)
AF
Exact solution
u
x


1D Heat conduction (N=80)
AF
Exact solution
u
x


1D Heat conduction (N=80)
AF
Exact solution
u
x

Error

Number of grid points	h	$\|\varepsilon\|_1$
N=20	5.000000000000000e-02	2.671748213469982E-003
N=40	2.500000000000000e-02	5.252468865364790E-004
N=80	1.250000000000000e-02	7.074975292198360E-005
N=160	6.250000000000000e-03	8.365114626940341E-006

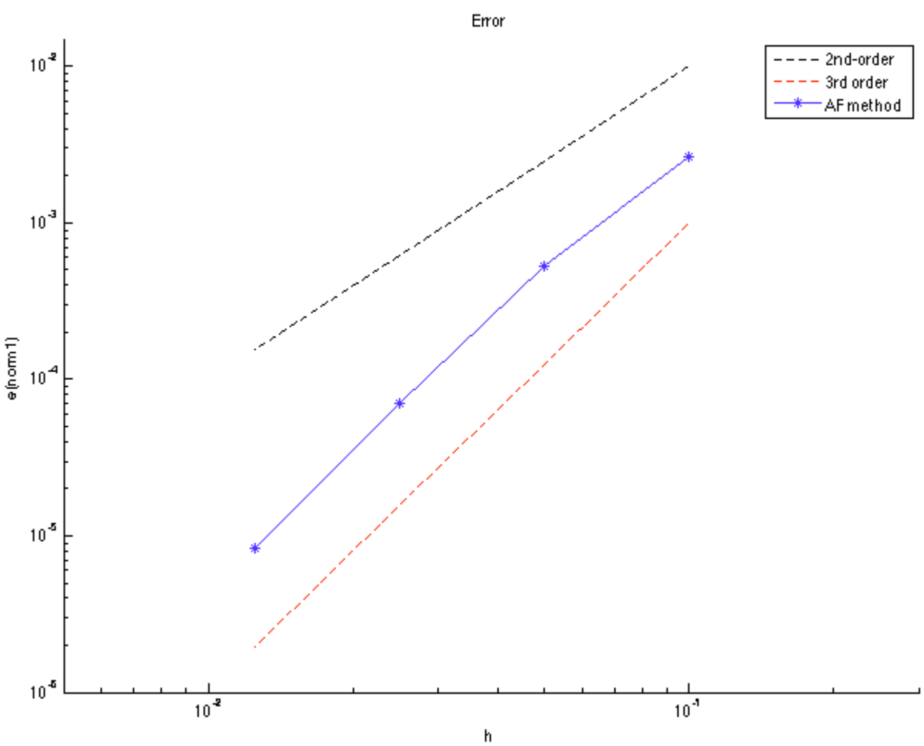


Fig. 1.4.3 Error

From the error shown above, the scheme is of 3rd order.